Numerical Solutions of Fourth-order Singular Boundary Value Problems by New Modified Adomian Decomposition Method

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Abstract

In this work, we apply Modified Adomian decomposition method (MADM) to solving fourth-order singular boundary value problems. The suggested method can be applied to linear and nonlinear problems. The planner is checked for some examples and the obtained results statement competence of the suggested method.

1 INTRODUCTION

This paper focuses on the following class of the 4th order singular boundary value problems (SBVP's),

$$\alpha y^{(4)} + \frac{n}{x} y^{(3)} + N(x)y'' + M(x)y' = g(x, y), 0 \leq x \leq 1, \tag{1}$$

with boundary condition

$$y(0) = \beta_1, y'(0) = \beta_2, y''(0) = \beta_3, y(a) = \beta_4, \tag{2}$$

where $\alpha = 1, n, \beta_i, i = 1, 2, 3, 4, a$ are constants and $N(x), M(x), g(x, y)$ are know functions linear or non-linear. The SBVP's problems arise in Mathematical modeling of several real life phenomena in different fields of study such as chemical reactions, electrodynamics, aerodynamics, thermal explosions, elastic stability, gravity assisted flows. Inelastic flows, atomic nuclear reactions and electrically charged fluid flows.

Recently, studies solving the SBVP's have introduced many analytical and numerical techniques. have been proposed for solving SBVP's. For example, Khuri has proposed a new decomposition method based on Adomian polynomials [1] for numerical treatment of generalized Lane-Emden type equations. In addition, Kim and Chun [2] suggested another MADM for series solution of higher order SBVP's. Moreover, in order to examine the power series solution of higher order SBVP's Aruna and Kanth [3] have employed differential transformation method. In another study, Wazwaz [4] investigated the approximate solution of fourth order initial value problems by means of variational iteration method. Taiwo and Hassan [5] introduced a new iterative decomposition method for solving higher order initial and boundary value problems. The authors in [6] studied the series solution of a class of fourth order singular initial value problems using the MADM. Hasan and Zhu [7, 8] have also proposed another MADM and applied it for solving singular boundary value problems of higher-order ordinary differential equations. In many other studies, the spline approximation techniques have widely been applied for numerical simulation of initial and boundary value problems (BVP's). For instance, in [9-13] the researchers applied the cubic spline (CS) functions for solving second order SBVP's. Moreover, Khuri and Sayfy [14] developed a new adaptive cubic B-spline (CBS) collocation approach for numerical solution of second order SBVP's. In addition, Mishra and Saini [15] explored the approximate solution of 3rd order self adjoint singularly perturbed BVP's applying typical QBS collocation method. In another study, Akram and Amin [16] have employed fifth degree polynomial spline functions for solving fourth order singularly perturbed BVP's. Lodhi and Mishra [17] applied quintic B-spline (QnBS) functions for numerical treatment of fourth order singularly perturbed SBVP's.

Our aim in this paper, we apply (MADM) to solving fourth-order singular boundary value problems.

2 THE NEW METHOD APPROXIMATION FOR $y^{(4)}(x)$

Re-write Eq.(1), as

$$L_A(.) + L_B(.) = g(x, y), \tag{3}$$

or

$$L_A(.) = g(x, y) - L_B(.), \tag{4}$$

or

$$L_B(.) = g(x, y) - L_A(.), \tag{5}$$

remembering that

$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x^{4-n}\frac{d}{dx}x^{n-3}(.), \tag{6}$$

$$L_B(.) = N(x)e^{-\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}e^{\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}(.). \tag{7}$$

If $L_B(.) = 0$, then $L_A(.)$ exist [7], where

$$L_A^{-1}(.) = x^{3-n}\int_a^x x^{n-4}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdx,$$

and

$$L_B^{-1}(.) = \int_0^x e^{-\int \frac{M(x)}{N(x)}dx}\int_0^x e^{\int \frac{M(x)}{N(x)}dx}N(x)^{-1}(.)dxdx.$$

Take L_A^{-1} on both sides Eq.(4), we have

$$y(x) = \gamma(x) + L_A^{-1}g(x, y) - L_A^{-1}L_B(y), \tag{8}$$

where

$$L(\gamma(x)) = 0,$$

the Adomain method give the solution $y(x)$ and function $g(x, y)$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \tag{9}$$

and

$$g(x, y) = \sum_{n=0}^{\infty} A_n, \tag{10}$$

$$L_B(y(x)) = N(x)e^{-\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}e^{\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}(\sum_{n=0}^{\infty} y_n(x)), \tag{11}$$

where $y_n(x)$ the elements are solution $y(x)$, and algorithms [18, 19] to formulate Adomian polynomials A_n. The following algorithm:

$$A_0 = S(y_0),$$

$$A_1 = y_1S'(y_0),$$

$$A_2 = y_2S'(y_0) + y_1^2\frac{1}{2}S''(y_0),$$

$$A_3 = y_3S'(y_0) + y_1y_2S''(y_0) + y_1^3\frac{1}{3!}S'''(y_0), \tag{12}$$

$$\dots$$

Substituting (9), (10) and (11) into (8), we get

$$\sum_{n=0}^{\infty} y_n(x) = \gamma(x) + L_A^{-1}\sum_{n=0}^{\infty} A_n - L_A^{-1}(N(x)e^{-\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}e^{\int \frac{M(x)}{N(x)}dx}\frac{d}{dx}(\sum_{n=0}^{\infty} y_n(x))), \quad (13)$$

the components $y_n(x)$ can be determined as

$$y_0(x) = \gamma(x),$$

$$y_{k+1}(x) = L_A^{-1}A_k - L_A^{-1}(L_B y_k), k \geq 0,$$

which gives

$$\begin{aligned} y_0(x) &= \gamma(x), \\ y_1(x) &= L_A^{-1}A_0 - L_A^{-1}(L_B y_0), \\ y_2(x) &= L_A^{-1}A_1 - L_A^{-1}(L_B y_1), \\ y_3(x) &= L_A^{-1}A_2 - L_A^{-1}(L_B y_2), \quad (14) \\ &\cdots \end{aligned}$$

We can determine the components $y_n(x)$, from (12) and (14), the series solution of $y(x)$ give

$$\Psi_n = \sum_{k=0}^{n-1} y_k,$$

can be used to approximate the exact solution.

3 NUMERICAL RESULTS

In the section, we give numerical results of new approximation method for solution $y^{(4)}$, giving illustrative examples to it.

3.1 Example

Consider the fourth-order singular boundary value problem:

$$y^{(4)} + \frac{3}{x}y^{(3)} + \frac{1}{x}y'' + y' = g(x) + y, \quad (15)$$

where

$$g(x) = \frac{-1}{2\,x} + \frac{1536}{(4+x^2)^4} - \frac{192}{(4+x^2)^3} + \frac{4\,x}{(4+x^2)^2} - \frac{3\,x}{2\,(4+x^2)} - \log(\frac{1}{4+x^2}),$$

with boundary condition

$$y(0.5) = -1.44692, y'(0) = 0, y''(0) = -0.5.$$

The true solution is

$$y = \log(\frac{1}{4+x^2}),$$

re-write Eq.(15), as

$$L_A(.) + L_B(.) = g(x) + y, \qquad (16)$$

where

$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x\frac{d}{dx}(.), \qquad (17)$$

and

$$L_B(.) = \frac{1}{x}e^{-x^2}\frac{d}{dx}e^{x^2}\frac{d}{dx}(.), \qquad (18)$$

re-write Eq.(16), as

$$L_A(.) = g(x) + y - L_B(.), \qquad (19)$$

where inverse differential operator for $L_A(.)$, we have

$$L_A^{-1}(.) = \int_{0.5}^{x} x^{-1}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdx,$$

applying $L_A^{-1}(.)$ to both sides of Eq.(19), we get

$$y(x) = -1.38442 - 0.25\,x^2 + L_A^{-1}(g(x) + y) - L_A^{-1}L_B(.),$$

we give

$$y_0 = -1.38442 - 0.25\,x^2 + L_A^{-1}g(x),$$

$$y_{n+1} = L_A^{-1}y_n - L_A^{-1}L_B(y_n), n \geq 0,$$

so that

$$y_0 = -1.38372 + 1.85727\,10^{-19}\,x - 0.25\,x^2 - 0.0277778\,x^3 + ... + 0.000194589\,x^{10},$$

$$y_1 = L_A^{-1}(y_0) - L_A^{-1}(L_B y_0),$$

remembering that

$$L_A^{-1}(.) = \int_{1}^{x} x^{-1}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdx,$$

$$L_B(y_0) = \frac{1}{x}e^{-x^2}\frac{d}{dx}e^{x^2}\frac{d}{dx}(y_0).$$

Now, we get

$$L_A^{-1} y_0 = 0.000906365 - 0.0144137\, x^4 - 0.000347222\, x^6 + ... + 6.75154\, 10^{-7}\, x^{10},$$

and

$$L_A^{-1} L_B(y_0) = -0.00363042 + 1.95156\, 10^{-18}\, x + 2.79182\, 10^{-18}\, x^2 + ... + 1.0334\, 10^{-8}\, x^{10},$$

we have

$$y_1 = -0.00272406 + 1.95156\, 10^{-18}\, x + 2.79182\, 10^{-18}\, x^2 + 0.0277778\, x^3 + ... + 6.85488\, 10^{-7}\, x^{10},$$

$$y_2 = 0.0000980296 - 2.61488\, 10^{-20}\, x - 1.75052\, 10^{-19}\, x^2 + ... + 3.04318\, 10^{-7}\, x^{10},$$

$$y(x) = y_0 + y_1 + y_2 = -1.38634 + 1.93922\, 10^{-18}\, x - 0.25\, x^2 + 2.71782\, 10^{-17}\, x^3 + 0.0312485\, x^4 + 0.00159615\, x^5 - 0.00523858\, x^6 - 0.0000995887\, x^7 + 0.000976129\, x^8 + 6.39685\, 10^{-6}\, x^9 - 0.00019497\, x^{10},$$

Table 1. Compare between the exact solution with the approximate MADM in [0,1].

x	Exact	MADN	Absolute
0.0	-1.38629	-1.38634	0.00005
0.1	-1.38879	-1.38884	0.00005
0.2	-1.39624	-1.39629	0.00005
0.3	-1.40854	-1.40859	0.00004
0.4	-1.42552	-1.42555	0.00003
0.5	-1.44692	-1.44692	0.00000
0.6	-1.47247	-1.47247	0.00000
0.7	-1.50185	-1.50164	0.00019
0.8	-1.53471	-1.53427	0.00044
0.9	-1.57070	-1.56987	0.00083
1.0	-1.60944	-1.60802	0.00142

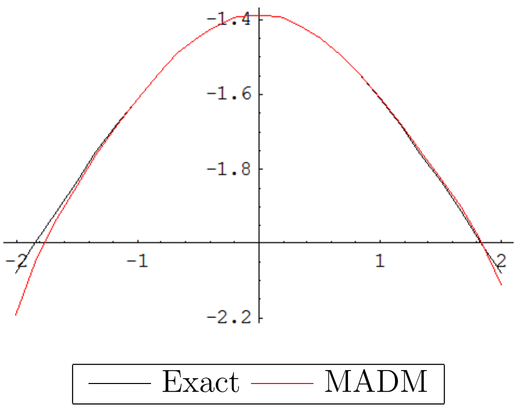


Figure 1: The Approximation for the exact solution and MADM.

In Table 1 and Figure 1, we noted that the solution very closed into the true solution. So the method is very efficient.

3.2 Example

Suppose the fourth-order Emden-Flower type equation [20,21]

$$y^{(4)} + \frac{3}{x}y^{(3)} = 96(1 - 10x^4 + 5x^8)e^{-4y}, \quad 0 \leq x \leq 1, \tag{20}$$

$$y'(0) = 0, y''(0) = 0, y(0.2) = 0.0016,$$

with the right solution is $y = \log\left(1 + x^4\right)$,

re-write Eq.(20), as

$$L_A(.) = 96(1 - 10x^4 + 5x^8)e^{-4y}, \tag{21}$$

where

$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x\frac{d}{dx}(.), \tag{22}$$

exist the operator in [7]

where inverse differential operator for $L_A(.)$, we have

$$L_A^{-1}(.) = \int_{0.2}^{x} x^{-1} \int_0^x \int_0^x \int_0^x x(.)dxdxdxdx.$$

Applying $L_A^{-1}(.)$ to both sides of Eq.(21), we get

$$y(x) = 0.0016 + L_A^{-1}(96(1 - 10x^4 + 5x^8)e^{-4y}),$$

we give

$$y_0 = 0.0016,$$

$$y_{n+1} = L_A^{-1}(96(1-10x^4+5x^8)e^{-4y_n}), n \geq 0,$$

then

$$y_1 = L_A^{-1}(96(1-10x^4+5x^8)e^{-4y_0}).$$

Now, we have

$$y_1 = -0.00158888 + 0.99362\,x^4 - 0.354864\,x^8 + 0.0301097\,x^{12},$$

$$y_2 = -1.55811\,10^{-8} + 0.000010104\,x^4 - 0.000229273\,x^8 + 0.000396928\,x^{12} - 0.0000970552\,x^{16} + 9.26172\,10^{-6}\,x^{20} - 3.15337\,10^{-7}\,x^{24},$$

$$y(x) = y_0+y_1+y_2 = 0.0000111001+0.993631\,x^4-0.355094\,x^8+0.0305066\,x^{12} - 0.0000970552\,x^{16} + 9.26172\,10^{-6}\,x^{20} - 3.15337\,10^{-7}\,x^{24},$$

Table 2. Compare between the exact solution with the approximate MADM in [0,1]

x	Exact	MADN	Absolute
0.0	0.000000	0.000011	0.000011
0.1	0.000095	0.000110	0.000010
0.2	0.001599	0.001600	0.000001
0.3	0.008067	0.008036	0.000031
0.4	0.025277	0.025215	0.000062
0.5	0.060625	0.060766	0.000109
0.6	0.121860	0.122889	0.001028
0.7	0.215192	0.218533	0.003341
0.8	0.343306	0.349521	0.006215
0.9	0.504465	0.507675	0.003210
1.0	0.693147	0.668966	0.024181

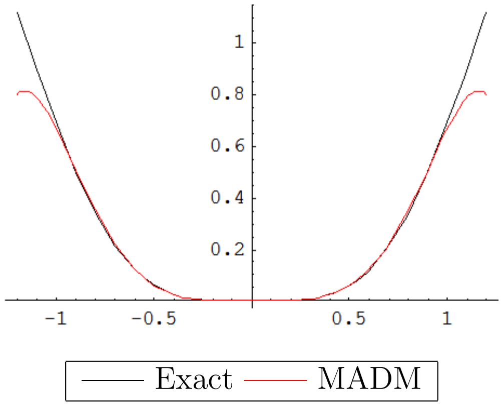


Figure 2: The Approximation for the exact solution and MADM.

The example 3.2, we see the approximation results for Figure 2. it is clear that the obtained results are good solution into the exact solution. So the new method for solving fourth-order by is effective.

3.3 Example

Consider the problem:

$$y^{(4)} + y'' + y' = 3e^x - x + Lny, \tag{23}$$

$$y(1) = 2.71828, y(0) = 1, y'(0) = 1, y''(0) = 1,$$

with the right solution is $y = e^x$,

re-write Eq.(23), as

$$L_A(.) + L_B(.) = 3e^x - x + Lny, \tag{24}$$

where

$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x^4\frac{d}{dx}x^{-3}(.), \tag{25}$$

and

$$L_B(.) = e^{-x}\frac{d}{dx}e^x\frac{d}{dx}(.), \tag{26}$$

re-write Eq.(24), as

$$L_A(.) = 3e^x - x + Lny - L_B(.), \tag{27}$$

where inverse differential operator for $L_A(.)$, we have

$$L_A^{-1}(.) = x^3\int_{0.2}^{x} x^{-4}\int_0^x\int_0^x\int_0^x x(.)dxdxdxdx,$$

applying $L_A^{-1}(.)$ to both sides of Eq.(27), we get

$$y(x) = 1. + 1.\,x + 0.5\,x^2 + 0.218282\,x^3 + L_A^{-1}(3e^x - x + Lny) - L_A^{-1}L_B(.),$$

we give

$$y_0 = 1. + 1.\,x + 0.5\,x^2 + 0.218282\,x^3 + L_A^{-1}(3e^x - x),$$
$$y_{n+1} = L_A^{-1}Lny_n - L_A^{-1}L_B(y_n), n \geq 0,$$

so that

$$y_0 = 1. + x + 0.5\,x^2 + 0.13914\,x^3 + 0.0416667\,x^4 + 0.0166667\,x^5 + 0.0125\,x^6 + 0.00535714\,x^7 + 0.00200893\,x^8 + 0.000669643\,x^9 + 0.000200893\,x^{10},$$
$$y_1 = L_A^{-1}(Lny_0) - L_A^{-1}(L_By_0),$$

remembering that

$$L_A^{-1}(.) = x^3 \int_{0.2}^{x} x^{-4} \int_0^x \int_0^x \int_0^x x(.)dxdxdxdx,$$
$$L_B(.) = e^{-x}\frac{d}{dx}e^x\frac{d}{dx}(.),$$

now, we get

$$L_A^{-1}Lny_0 = -0.00831632\,x^3 + 2.1684\,10^{-18}\,x^4 + 0.00833333\,x^5 + 4.06576\,10^{-19}\,x^6 - 0.0000327704\,x^7 + 0.0000163852\,x^8 - 1.79571\,10^{-6}\,x^9 + 1.38626\,10^{-6}\,x^{10},$$

and

$$-L_A^{-1}L_B(y_0) = 0.102179\,x^3 - 0.0833333\,x^4 - 0.0152903\,x^5 - 0.00254839\,x^6 - 0.000595238\,x^7 - 0.000272817\,x^8 - 0.0000992063\,x^9 - 0.0000297619\,x^{10},$$

we have

$$y_1 = L_A^{-1}Lny_0 - L_A^{-1}L_B(y_0)$$
$$= 0.093863\,x^3 - 0.0833333\,x^4 - 0.00695698\,x^5 - 0.00254839\,x^6 - 0.000628008\,x^7 - 0.000256432\,x^8 - 0.000101002\,x^9 - 0.0000283756\,x^{10},$$
$$y_2 = 0.00201143\,x^3 + 6.77626\,10^{-20}\,x^4 - 0.00469315\,x^5 + 0.00199559\,x^6 + 0.000674209\,x^7 - 0.0000392618\,x^8 + 0.0000545551\,x^9 - 6.26227\,10^{-6}\,x^{10},$$

the solution give by

$$y(x) = y_0 + y_1 + y_2 = 1. + x + 0.5\,x^2 + 0.235014\,x^3 - 0.0416667\,x^4 + 0.00501654\,x^5 + 0.0119472\,x^6 + 0.00540334\,x^7 + 0.00171323\,x^8 + 0.000623196\,x^9 + 0.000166255\,x^{10},$$

Table 3. Compare between the exact solution with the approximate MADM in [0,1].

x	Exact	MADN	Absolute
0.0	1.00000	1.00000	0.00000
0.1	1.10517	1.10523	0.00006
0.2	1.22140	1.22182	0.00042
0.3	1.34986	1.35103	0.00217
0.4	1.49182	1.49408	0.00316
0.5	1.64872	1.65217	0.00345
0.6	1.82212	1.82650	0.00438
0.7	2.01375	2.01843	0.00468
0.8	2.22554	2.22956	0.00402
0.9	2.45961	2.46192	0.00231
1.0	2.71828	2.71821	0.00007

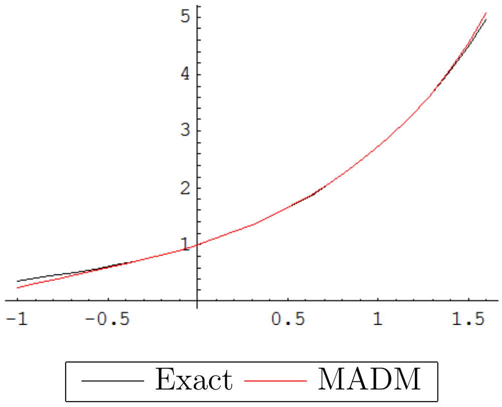


Figure 3: The Approximation for the exact solution and MADM.

We noticed that the approximate solution by MADM to reach the right solution when we collected y_0, y_1, y_2, we get the approximate solution but if we continue to y_n, we will get the right solution.

3.4 Example

Consider the problem:

$$y^{(4)} + y'' + y' = 2 + 2x - x^4 + y^2, \tag{28}$$

$$y(0) = 0, y'(0) = 0,$$

with the exact solution is $y = x^2$,

re-write Eq.(28), as

$$L_A(.) + L_B(.) = 2 + 2x - x^4 + y^2, \tag{29}$$

where

$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x^4\frac{d}{dx}x^{-3}(.), \tag{30}$$

and

$$L_B(.) = e^{-x}\frac{d}{dx}e^x\frac{d}{dx}(.), \tag{31}$$

re-write Eq.(29), as

$$L_B(.) = 2 + 2x - x^4 + y^2 - L_A(.), \tag{32}$$

where inverse differential operator for $L_B(.)$, we have

$$L_B^{-1}(.) = \int_0^x e^{-x}\int_0^x e^x(.)dxdx,$$

Applying $L_B^{-1}(.)$ to both sides of Eq.(32), we get

$$y(x) = L_B^{-1}(2 + 2x - x^4 + y^2) - L_B^{-1}L_A(.),$$

we give

$$y_0 = 0 + L_B^{-1}(2 + 2x - x^4),$$
$$y_{n+1} = L_B^{-1}y_n^2 - L_B^{-1}L_A(y_n), n \geq 0,$$

so that

$$y_0 = x^2 - \frac{x^6}{30} + \frac{x^7}{210} - \frac{x^8}{1680} + \frac{x^9}{15120} - \frac{x^{10}}{151200},$$

remembering that

$$L_B^{-1}(.) = \int_0^x e^{-x}\int_0^x e^x(.)dxdx,$$
$$L_A(.) = x^{-1}\frac{d^3}{dx^3}x^4\frac{d}{dx}x^{-3}(.),$$

now, we get

$$L_B^{-1}y_0^2 = \frac{x^6}{30} - \frac{x^7}{210} + \frac{x^8}{1680} - \frac{x^9}{15120} - \frac{37\,x^{10}}{50400},$$

and

$$L_A(y_0) = -12\,x^2 + 4\,x^3 - x^4 + \frac{x^5}{5} - \frac{x^6}{30} + \frac{x^7}{210} - \frac{x^8}{1680} + \frac{x^9}{15120} - \frac{x^{10}}{151200},$$

$$-L_B^{-1}L_A(y_0) = x^4 - \frac{2\,x^5}{5} + \frac{x^6}{10} - \frac{2\,x^7}{105} + \frac{x^8}{336} - \frac{x^9}{2520} + \frac{x^{10}}{21600},$$

we have

$$y_1 = L_B^{-1}y_0^2 - L_B^{-1}L_A(y_0) =$$

$$y_1 = x^4 - \frac{2\,x^5}{5} + \frac{2\,x^6}{15} - \frac{x^7}{42} + \frac{x^8}{280} - \frac{x^9}{2160} - \frac{13\,x^{10}}{18900},$$

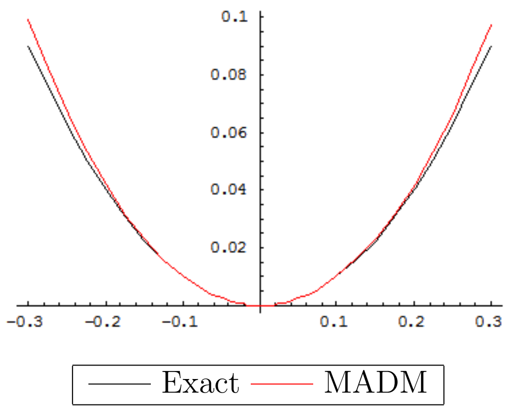
the solution give by

$$y(x) = y_0 + y_1 = x^2 + x^4 - \frac{2\,x^5}{5} + \frac{x^6}{10} - \frac{2\,x^7}{105} + \frac{x^8}{336} - \frac{x^9}{2520} - \frac{x^{10}}{1440},$$

Table 4. Compare between the exact solution with the approximate MADM in [-0.3,0.3]

x	Exact	MADN	Absolute
-0.3	0.09	0.099149	0.009149
-0.2	0.04	0.041735	0.001735
-0.1	0.01	0.010104	0.000104
0.0	0.00	0.0000000	0.000000
0.1	0.01	0.0100961	0.000096
0.2	0.04	0.0414782	0.001478
0.3	0.09	0.0971969	0.037197

Figure 4: The Approximation for the exact solution and MADM.



4 Conclusion

In this work, we used the a new MADM for solving SBVP's. We have demonstrated that the method is quick convergent for solving SVP's. The given examples illustrate the advantages of using the proposed method in this work for these kinds of equations. Finally the Modified Adomian decomposition method is effective in finding the numerical solutions for a wide class of boundary value problems.

5 References

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