Multipliers in semihoops

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Abstract

Semihoops play an important role in the study of fuzzy logic based on left continuous t-norms. In this paper, we introduce the notion of multipliers in semihoops and investigate some related properties of them. Also, we discuss the relations between multipliers and closure operators in semihoops. Moreover, we focus on algebraic structures of the set $IM(L)$ of all implicative multipliers in semihoops and obtain that $IM(L)$ forms a Heyting algebra, when L is an MTL-algebra.

Keywords: Semihoop; multipliers; closure operator

1. Introduction

Much of human reasoning and decision making is based on an environment of imprecision, uncertainty, incompleteness of information, partiality of truth and partiality of possibility-in short, on an environment of imperfect information. Hence how to represent and simulate human reasoning become a crucial problem in information science field. For this reason, various logical algebras have been proposed as the semantical systems of non classical logic systems, for example, MV-algebras, BL-algebras, MTL-algebras, residuated latticese, hoops and semihoops. Among these logical algebras, semihoops [1] are very basic algebraic structures and contain all logical algebras based on residuated lattices. Semihoops are generalizations of hoops which were introduced by Bosbach. In the last few years, the theory of hoops has been enriched with deep structure theorems[2, 3, 4, 5, 6, 7]. Many of these results have a strong impact with fuzzy logics. In particular, from the structure theorem of finite basic hoops, one obtains an elegant short proof of the completeness theorem for propositional basic logic, which introduced by Hájek [8]. As a more general structure, a semihoop is a hoop without the condition $x \odot (x \to y) = y \odot (y \to x)$. It follows that a semihoop does not satisfy the divisibility condition $x \wedge y = x \odot (x \to y)$. Compared to hoops contains all algebraic structures that induce by continuous t-norms [10], semihoops contains all algebraic structures that induce by left continuous t-norms. Therefore, semihoops play an important role in studying fuzzy logics and the related algebraic structures.

The notion of multipliers, introduced from the analytic theory, is helpful for studying algebraic structures and properties in algebraic systems. Multipliers in a commutative semigroup $(A, *)$ were introduced by Larsen[11] , which was defined by a function f from A into A such that $f(x) * y = x * f(y)$ for all $x, y \in A$. Consequently, the notion of multipliers has been extended to distributive lattices[12, 13], BE-algebras[14], d-algebras[15] and BL-algebras[16]. In particular, A. Borumand Saeid[16] introduced a multiplier in BL-algebras L by a function f from L into L such that $f(x \to y) = x \to f(y)$ for all $x, y \in L$ and used multipliers to study the algebraic structures of MV-center of BL-algebras. As we have mentioned in the above, obstinate fifilters have been widely studied on BL-algebras, residuated lattices and MV-algebras, etc. All the above mentioned algebraic structures are the special case of semihoops. In fact, semihoops are the widest possible residuated structure. Therefore, it is interesting to study the multipliers on semihoops for providing a more general algebraic foundation for inference rule in fuzzy logic based on left continuous t-norms. This is the motivation for us to investigate multipliers on semihoops.

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2. Preliminaries

In this section, we summarize some definitions and results about semihoops which will be used in the following sections.

Definition 2.1. [2, 10] An algebra $(L, \wedge, \odot, \rightarrow, 1)$ of type (2,2,2,0) is called a *semihoop* if it satisfies the following conditions:

(1) $(L, \wedge, 1)$ is a $\wedge$-semilattice with upper bounded 1,
(2) $(L, \odot, 1)$ is a commutative monoid,
(3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$, for all $x, y, z \in L$.

In what follows, for any $x \in L$, we define $x^0 = 1$ and $x^n = x^{n-1} \odot x$ for any natural number n.

On a semihoop L, we define $x \leq y$ if and only if $x \rightarrow y = 1$ for all $x, y \in L$. It is easy to check that $\leq$ is a partial order relation on L and for all $x \in L$, $x \leq 1$. Moreover, an algebra L is a bounded semihoop if L is a semihoop and there exists an element $0 \in L$ such that $0 \leq x$ for all $x \in L$. In a bounded semihoop L, we define the negation $* : x^* = x \rightarrow 0$ for all $x \in L$. If $x \odot x = x$, that is, $x^2 = x$ for all $x \in L$, then the semihoop L is said to be idempotent. It is easy to check that an idempotent semihoop is equivalent to a Brouwerian semilattice [17]. In this work, unless mentioned otherwise, $(L, \wedge, \odot, \rightarrow, 0, 1)$ will be a bounded semihoop, which will often be referred by its support set L.

Proposition 2.2. [2, 3, 5, 6, 7, 9] In any semihoop L, the following properties hold: for any $x, y, z \in L$,

(1) $x \leq y \rightarrow x$,
(2) $x \rightarrow 1 = 1$,
(3) $1 \rightarrow x = x$,
(4) $x \leq y \Rightarrow x \rightarrow z \geq y \rightarrow z$,
(5) $x \leq y \Rightarrow z \rightarrow x \leq z \rightarrow y$,
(6) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
(7) $x \odot y \leq z$ iff $x \leq y \rightarrow z$,
(8) $x \odot y \leq x, y$,
(9) $x \odot y \leq x \wedge y$,
(10) $x \rightarrow y \leq (z \rightarrow x) \rightarrow (z \rightarrow y)$,
(11) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
(12) $(x \rightarrow y) \odot (y \rightarrow z) \leq x \rightarrow z$.

Proposition 2.3. [2, 3] In any bounded semihoop L, the following properties hold: for any $x, y, z \in L$,

(1) $0^* = 1$, $1^* = 0$,
(2) $x \leq y \Rightarrow x^* \geq y^*$,
(3) $x \odot x^* = 0$.

3. Multipliers in semihoops

In the section, we introduce the notion of implicative multipliers in semihoops and investigate some related properties of such operators. Also the algebraic structure of the set $IM(L)$ of all implicative multiplier in semihoops be studied.

Definition 3.1. *Let L be a semihoop. A mapping $f : L \rightarrow L$ is called a implicative multiplier on L if it satisfies the following condition:*

$$f(x \rightarrow y) = x \rightarrow f(y)$$

For any implicative multiplier f on L, the *kernel* of f is the set $Ker(f) = \{x \in L | f(x) = 1\}$. f is called *faithful* if $Ker(f) = 1$.

Now, we present some examples of implicative multipliers in semihoops.

Example 3.2.

(1) Obvious, id_L is a implicative multiplier in semihoop L.
(2) Let L be a semihoop and $f(x) = 1$, for any $x \in L$. Then f is a implicative multiplier in L. We denoted this mapping by 1_f.
(3) $f_p(x) = p \to x$ is a implicative multiplier in every semihoop L, where $p \in L$. $f_p(x)$ is called the principle implicative multiplier in L.
(4) Let L_1, L_2 be two semihoops. Then $L_1 \times L_2$ is also a semihoop w.r.t. the point-wise operations (such as :$(a, b) \to (c, d) = (a \to c, b \to d)$). If we define two maps $f, g : L_1 \times L_2 \to L_1 \times L_2$ by $f(x, y) = (x, 1)$ and $g(x, y) = (1, y)$, for any $(x, y) \in L_1 \times L_2$. One can easily check that f and g are implicative multipliers in $L_1 \times L_2$ w.r.t. the point-wise operations.
(5) Let $L = \{0, a, b, 1\}$, $0 < a < b < 1$ and $\odot, \to$ define as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L,\wedge,\odot,\to,0,1)$ is a semihoop. Now, we define a map f on L as follows:

$$f(x) = \begin{cases} a, & x = 0 \\ b, & x = b \\ 1, & x = a, 1 \end{cases}.$$

We have f is a implicative multiplier in L but not faithful.

Next, we present some properties of implicative multipliers in semihoops.

Proposition 3.3. *Let f be a implicative multiplier in semihoop L. Then the follows hold: for any $x, y \in L$,*

(1) $f(1) = 1$,
(2) $x \le y$ *implies* $x \le f(y)$,
(3) $x \le f(x)$,
(4) $f(x) \to y \le x \to y \le x \to f(y)$ *and* $f(x) \to y \le f(x) \to f(y) \le x \to f(y)$,
(5) $(f(x))^* \le x^* \le f(x^*)$. *In particular, if* $f(0) = 0$, *then* $f(x^*) = x^*$,
(6) *if f is faithful, then* $f(x) = x$.

Proof.

(1) Obvious, $0 \to x = 1$ for any $x \in L$. Then $f(1) = f(0 \to x) = 0 \to f(x) = 1$ implies $f(1) = 1$.
(2) If $x \le y$, then $x \to y = 1$, and hence $f(x \to y) = x \to f(y) = 1$, which implies $x \le f(y)$.
(3) It is straightforward from item (2).
(4) From Proposition 3.3(3), we have $f(x) \to y \le x \to y \le x \to f(y)$ and $f(x) \to y \le f(x) \to f(y) \le x \to f(y)$.
(5) It is straightforward from item (4).
(6) Since $f(f(x) \to x) = f(x) \to f(x) = 1$ and f is faithful, we have $f(x) \to x = 1$ implies $f(x) \le x$. Together with item (3), thus $f(x) = x$.

Theorem 3.4. *Let f be a implicative multiplier in semihoop L. Then the follow statements are equivalent:*

(1) $f(x) \to x = 1$;
(2) *f is an identity mapping;*
(3) *f satisfying the following conditions:*
(i) $f^2 = f$,
(ii) $f(x \to y) = f(x) \to f(y)$,
(iii) $f^2(x) \to y = f(x) \to f(y)$;
(4) $f(x) \to y = x \to f(y)$;
(5) *f is faithful.*

Proof. The equivalence between (1) and (2) are obvious.

(2) $\Rightarrow$ (3) Obviously.

(3) $\Rightarrow$ (4) $x \to f(y) = f(x \to y) = f(x) \to f(y) = f^2(x) \to y = f(x) \to y$.

(4) $\Rightarrow$ (2) Let $x = 1$. Then $f(1) \to y = 1 \to f(y)$, i.e., $f(y) = y$.

From Proposition 3.3(6), the equivalence between (2) and (5) are obvious.

Definition 3.5. *Let f be a implicative multiplier in semihoop L. Then f is called:*

(1) *an isotone implicative multiplier if $x \leq y$ implies $f(x) \leq f(y)$, for any $x, y \in L$.*
(2) *an idempotent implicative multiplier if $f(f(x)) = f(x)$ (that is, $f^2 = f$), for any $x \in L$.*

Example 3.6.

(1) Let $L = \{0, a, b, c, 1\}$ be a chain, where $0 < a < b < c < 1$. Define operations $\odot$ and $\to$ as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	a	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	1	1
c	0	b	b	1	1
1	0	a	b	c	1

Then $(L, \wedge, \odot, \to, 0, 1)$ is a semihoop that is not divisible (because $b = b \wedge c \neq c \odot (c \to b) = c \odot b = a$). Now, we define a map f on L as follows:

$$f(x) = \begin{cases} 0, & x = 0 \\ b, & x = a, b \\ 1, & x = 1, c \end{cases}.$$

One can easily check that f is an isotone implicative multiplier in L. Also, f is idempotent.

(2) Let $L = \{0, a, b, 1\}$, $0 < a < b < 1$ and $\odot, \to$ define as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L,\wedge,\odot,\to,0,1)$ is a semihoop. One can easily check that the map $f_a(x) := a \to x$ on L is an isotone implicative multiplier, but not idempotent.

In fact, there exist implicative multipliers that are not isotone. For example, the multipliers in Example 3.2(5).

Proposition 3.7. *Let f be a implicative multiplier in semihoop L and f preserves $\to$. Then f is isotone.*

Proof. If f is a preserves implicative multiplier in semihoop L and $x \leq y$, then $1 = f(x \to y) = f(x) \to f(y)$, that is, $f(x) \leq f(y)$. Thus f is isotone.

In general, the convert of Proposition 3.7 is not true. For example, assume L is a semihoop of Example 3.6(2). Then $f_a(x) = a \to x$ is an isotone implicative multiplier in semihoops L. Put $x = a, y = 0$, then $f_a(a \to 0) = f_a(a) = a \to a = 1$. Meanwhile, $f_a(a) \to f_a(0) = (a \to a) \to (a \to 0) = 1 \to a = a$. Thus $f_a(a \to 0) \neq f_a(a) \to f_a(0)$.

Proposition 3.8. *Let f be a closure operator in semihoop L and f preserves $\to$. Then f is a implicative multiplier in L.*

Proof. Since f is preserves $\rightarrow$, together with Proposition 3.3(3), we obtain that $f(x \rightarrow y) = f(x) \rightarrow f(y) \leq x \rightarrow f(y)$. On the other hand , $x \rightarrow f(y) \leq f(x \rightarrow f(y)) = f(x) \rightarrow f^2(y) = f(x) \rightarrow f(y) = f(x \rightarrow y)$. Therefore f is a implicative multiplier in L.

Proposition 3.9. *Let f be an isotone implicative multiplier in semihoop L and $f^2 \leq f$. Then f is a closure operator on L.*

Proof. From Proposition 3.3(3) and $f^2 \leq f$, we have $f^2 = f$. Since f is an isotone multiplier, we obtain that f is a closure operator on L.

We denote the set of all implicative multipliers in semihoop L by $IM(L)$. Let $f_1, f_2 \in IM(L)$. Then we define $f_1 \sqcap f_2 : L \rightarrow L$ by $(f_1 \sqcap f_2)(x) = f_1(x) \wedge f_2(x)$, $f_1 \sqcup f_2 : L \rightarrow L$ by $(f_1 \sqcup f_2)(x) = f_1(x) \vee f_2(x)$, $f_1 \circ f_2 : L \rightarrow L$ by $(f_1 \circ f_2)(x) = f_1(f_2(x))$, $f_1 \leq f_2$ by $f_1(x) \leq f_2(x)$ for any $x \in L$. Therefore the following results hold.

Theorem 3.10. *Let f_1, f_2 are two implicative multipliers in semihoop L. Then*

(1) *$f_1 \circ f_2$ is a implicative multiplier in L;*
(2) *$f_1 \sqcap f_2$ is a implicative multiplier in L;*
(3) *$(M(L), \circ, id_L)$ is a monoid, where id_L is an identity mapping;*
(4) *L is an MTL-algebra implies $f_1 \sqcup f_2$ is a implicative multiplier in L.*

Proof.

(1) Since $(f_1 \circ f_2)(x \rightarrow y) = f_1(x \rightarrow f_2(y)) = x \rightarrow (f_1 \circ f_2)(y)$, we have $f_1 \circ f_2$ is a implicative multiplier in L.
(2) Since $(f_1 \sqcap f_2)(x \rightarrow y) = f_1(x \rightarrow y) \wedge f_2(x \rightarrow y) = (x \rightarrow f_1(y)) \wedge (x \rightarrow f_2(y)) = x \rightarrow f_1(y) \wedge f_2(y) = x \rightarrow (f_1 \sqcap f_2)(y)$, which implies $f_1 \sqcap f_2$ is a implicative multiplier in L.
(3) Obviously.
(4) If L is an MTL-algebra, then $(f_1 \sqcup f_2)(x \rightarrow y) = f_1(x \rightarrow y) \vee f_2(x \rightarrow y) = (x \rightarrow f_1(y)) \vee (x \rightarrow f_1(y)) = x \rightarrow f_1(y) \vee f_2(y) = x \rightarrow (f_1 \sqcup f_2)(y)$, thus $f_1 \sqcup f_2$ is a implicative multiplier in L.

Theorem 3.11. *Let L be an MTL-algebra. Then $(IM(L), \sqcap, \sqcup, \leq, \hookrightarrow, id_L, 1_f)$ forms a Heyting algebra (where $(f_1 \hookrightarrow f_2)(x) := \sqcup\{f | f_1 \sqcap f \leq f_2\}$).*

Proof. Firstly, we show that $(IM(L), \sqcap, \sqcup, id_L, 1_f)$ is a bounded distributive lattice with id_L as the the smallest element and 1_f as the greatest element. For any $f_1(x), f_2(x), f_3(x) \in IM(L)$, together with Theorem 3.10(2) and (4), we have $f_1 \sqcup (f_2 \sqcap f_3)$ and $(f_1 \sqcup f_2) \sqcap (f_2 \sqcup f_3)$ are implicative multipliers in L. Moreover, $(f_1 \sqcup (f_2 \sqcap f_3))(x \rightarrow y) = f_1(x \rightarrow y) \sqcup (f_2(x \rightarrow y) \sqcap f_3(x \rightarrow y)) = (x \rightarrow f_1(y)) \vee ((x \rightarrow f_2(y)) \wedge (x \rightarrow f_3(y))) = [(x \rightarrow f_1(y)) \vee (x \rightarrow f_2(y))] \wedge [(x \rightarrow f_1(y)) \vee (x \rightarrow f_3(y))] = [f_1(x \rightarrow y) \vee f_2(x \rightarrow y)] \wedge [f_1(x \rightarrow y) \vee f_3(x \rightarrow y)] = [(f_1 \sqcup f_2)(x \rightarrow y)] \wedge [(f_1 \sqcup f_3)(x \rightarrow y)] = [(f_1 \sqcup f_2) \sqcap (f_1 \sqcup f_3)](x \rightarrow y)$. Meanwhile, from Examples 3.2(1) and (2), we have that id_L and 1_f are implicative multipliers in semihoops. Together with Proposition 3.3(3), it is easily obtain that $id_L \leq f \leq 1_f$ for any $f \in IM(L)$, namely, id_L is the the smallest element and 1_f is the greatest element of $IM(L)$. Thus $IM(L), \sqcap, \sqcup, \leq, id_L, 1_f)$ form a bound distributive lattice .

The following will check that for any multipliers $m, f, g \in IM(L)$, $m \sqcap f \leq g$ if and only if $m \leq f \hookrightarrow g$. Obvious, $(f_1 \hookrightarrow f_2)(x) := \sqcup\{f | f_1 \sqcap f \leq f_2\}$ is well define. If $m \sqcap f \leq g$, then $m(x) \wedge f(x) \leq g(x)$ for any $x \in L$, that implies $m \in \{q | f \sqcap q \leq g\}$. So $m \leq \sqcup\{q | f \sqcap q \leq g\}$, namely, $m \leq f \hookrightarrow g$. Conversely, if $m \leq f \hookrightarrow g$, then $m(x) \leq (f \hookrightarrow g)(x)$ for any $x \in L$. Thus $m \leq \sqcup\{q | f \sqcap q \leq g\}$ implies $f \sqcap m \leq g$, that is, $m \sqcap f \leq g$. Therefore $(IM(L), \sqcap, \sqcup, \leq, \hookrightarrow, id_L, 1_f)$ forms a Heyting algebra.

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