METHOD OF SYSTEMATIC INSPECTION FOR SOLVING DIFFERENTIAL EQUATIONS

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ABSTRACT

The method of systematic inspection solves or helps in discovering the behavior of differential equations. This method solves differential equations by creating functions of the independent variable(s) from two opposite small functions called seeds. Starting with a seed you can build a bigger function. Adding the two opposite functions together will give a function that satisfies the differential equation.

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1. Introduction

no previous research was carried out regarding systematic inspection. As a matter of fact inspecting a solution for a differential equation or for a prticular integral was a matter of trial and error. In this paper I tried to make inspection a systematic method. It is applied to linear and non-linear differential equations. Using this method doesn't require much knowledge of differential equations. I adopted examples more than theory in this paper to clear up the concept. I solved simple examples although the method applies to complex ones.

2. How to inspect the solution

The procedure followed to inspect functions that satisfy an ordinary differential equation is the same as that for inspecting functions for a partial differential equation. Looking at the following example you will understand how to inspect a solution for an ordinary differential equation.

2.1 Example

Inspect the complementary function for the following differential equation :

$$\frac{dy}{dx} + y = 0$$

steps

1-put the two terms in two rows

$y =$

$\frac{dy}{dx} =$

2- since $\frac{dy}{dx} + y = x^n - x^n$ then put x^n opposite to y and $- x^n$ opposite to $\frac{dy}{dx}$ like this

$y = +x^n$

$\frac{dy}{dx} = -x^n$

$sum = 0$

this is valid for linear differential equations only.

$+ x^n$ is called positive seed and,

$- x^n$ is called negative seed.

They are called seeds because using them you can build two larger opposite functions as you will see soon

3- to simplify work , let us break step (2) into two cases and work them out separately.

First case

$y = +x^n$

$\frac{dy}{dx} =$

$sum = +x^n$

Second case

$y =$

$\frac{dy}{dx} = -x^n$

$sum = -x^n$

sum of both cases $= 0$

4- now we start the process of what I call *calculate and balance* for the first case.

A – calculate

$$y = +x^n$$

$$\frac{dy}{dx} = nx^{n-1}$$ by differentiating once

$sum = x^n + nx^{n-1}$

B – balance 1

$$y = +x^n - nx^{n-1}$$

$$\frac{dy}{dx} = nx^{n-1}$$ move it with an opposite sign

$sum = x^n$

C – calculate

$$y = +x^n - nx^{n-1}$$

$$\frac{dy}{dx} = nx^{n-1} - n(n-1)x^{n-2}$$ by differentiating once

$sum = x^n - n(n-1)x^{n-2}$

D – balance 2

$$y = +x^n - nx^{n-1} + n(n-1)x^{n-2}$$

$$\frac{dy}{dx} = nx^{n-1} - n(n-1)x^{n-2}$$

move with an opposite sign

$sum = x^n$

continuing the process of calculate and balance we get a series like this,

$y_+ = x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + n(n-1)(n-2)(n-3)x^{n-4}$ ... (1)

now we process the second case

$y =$

$\frac{dy}{dx} = -x^n$

A – calculate

$$y = -\frac{x^{n+1}}{n+1}$$

$$\frac{dy}{dx} = -x^{n}$$

by integrating once

B – balance 1

$$y = -\frac{x^{n+1}}{n+1}$$

$$\frac{dy}{dx} = -x^{n} + \frac{x^{n+1}}{n+1}$$

move with an opposite sign

C – calculate

$$y = -\frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{(n+1)(n+2)}$$

$$\frac{dy}{dx} = -x^{n} + \frac{x^{n+1}}{n+1}$$

by integrating once

D – balance 2

$$y = -\frac{x^{n+1}}{n+1} + \frac{x^{n+2}}{(n+1)(n+2)}$$

$$\frac{dy}{dx} = -x^{n} + \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{(n+1)(n+2)}$$

move with an opposite sign

repeat the process of calculate and balance till you get a satisfactory number of terms for y_- as you see below,

$$y_- = -\frac{x^{n+1}}{(n+1)} + \frac{x^{n+2}}{(n+1)(n+2)} - \frac{x^{n+3}}{(n+1)(n+2)(n+3)} + \frac{x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \ldots (2)$$

so the result is :

$$y = y_+ + y_-$$

$$y = x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + n(n-1)(n-2)(n-3)x^{n-4} \ldots$$

$$-\frac{x^{n+1}}{(n+1)} + \frac{x^{n+2}}{(n+1)(n+2)} - \frac{x^{n+3}}{(n+1)(n+2)(n+3)} + \frac{x^{n+4}}{(n+1)(n+2)(n+3)(n+4)} \ldots$$

this is the inspected function form from which we can get a function that satisfies the given differential equation , and it is convergent for $n \geq 0$.

By substituting different values of n in the above form, we note that only one function satisfies the differential equation and the others will be similar to it as we'll see soon, but if we have a second order differential equation then we will get two functions and the others will be similar to any of them, and three for third order and so on. (what I mean by similar is that the new function will be a previous inspected function multiplied by a constant). Now if we substitute $n = 0$ into the inspected function form above we get,

$y = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120} \ldots$ (this is Maclaurin's series for e^{-x}.)

all other inspected functions that
we can get for $n = 1,2,3,4,\ldots$ will be similar to the above inspected function. For example if you get the inspected function for $n = 1$ it will be:

$$y = -1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^5}{120} \ldots$$

which is equal to the inspected function for $n = 0$ multiplied by -1

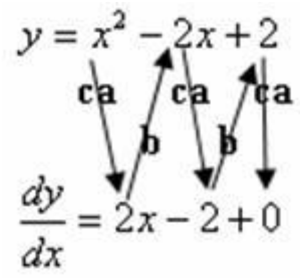
if you try other values of n you will get similar functions.

2.2 Finding the particular integral

Suppose that our differential equation looks like this,

$$\frac{dy}{dx} + y = x^2$$

then the particular integral can be obtained by putting x^2 opposite to y as follows,



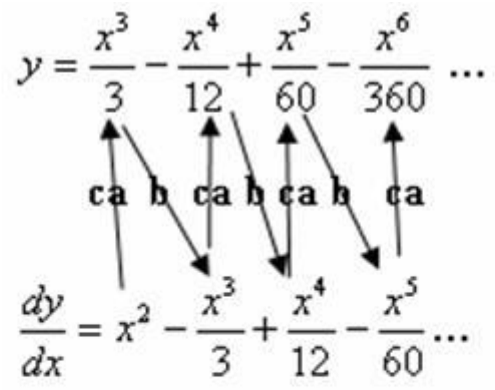
where ca means calculate and b means balance .

in the case of every calculation we differentiate once with respect to x and with balance we move with an opposite sign.

so particular integral is

$$y_p = x^2 - 2x + 2$$

we can get another particular integral by putting x^2 opposite to $\frac{dy}{dx}$, but since it is divergent so it is neglected,



in the case of every calculation we integrate once with respect to x .

then the general solution for the differential equation :

$$y_g = y_c + y_p$$

$$y_g = C(1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} \dots) + x^2 - 2x + 2$$

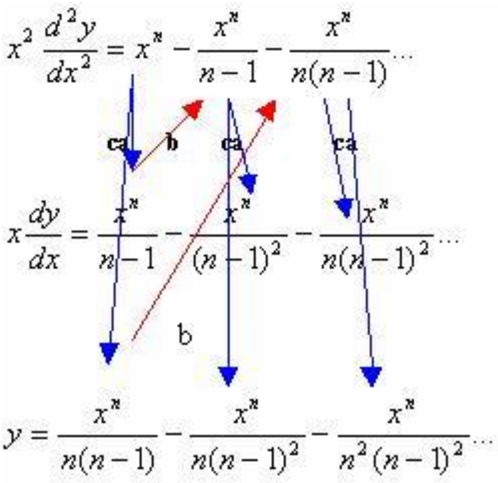
2.3 solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$$

solution

1- positive seed

put x^n opposite to $x^2 \frac{d^2y}{dx^2}$



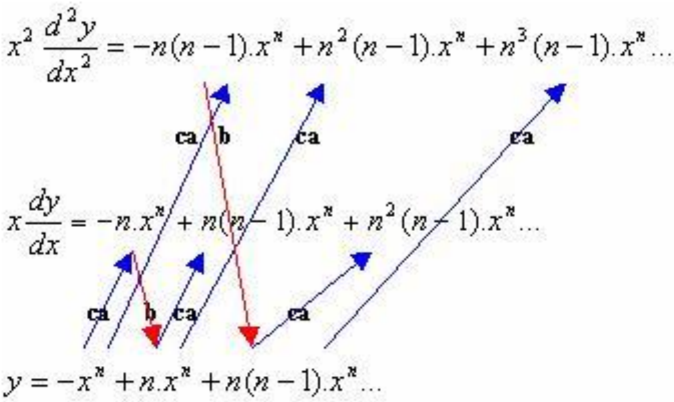
in the case of calculation from $x^2\frac{d^2y}{dx^2}$ to $x\frac{dy}{dx}$ divide by x^2 then calculate $\frac{dy}{dx}$ by integrating once with respect to x then multiply the result by x.

in the case of calculation from $x^2\frac{d^2y}{dx^2}$ to y divide by x^2 then calculate y by integrating twice with respect to x.

if we proceed more, the power of x will remain unchanged, that is x^n.

2- negative seed

put $-x^n$ opposite to y



in the case of calculation from y to $x\frac{dy}{dx}$ differentiate once and multiply by x.

in the case of calculation from y to $x^2 \frac{d^2y}{dx^2}$ differentiate twice and multiply by x^2.

since the resulting inspected function is a function of x^n, then $y = x^n$

substituting in the differential equation we get

$$x^2.n(n-1).x^{n-2} + x.n.x^{n-1} + x^n = 0$$

minimizing

$$x^n(n^2 - n + n + 1) = 0$$

$$n^2 + 1 = 0$$

$$(n-i)(n+i) = 0$$

which gives us two inspected functions :

$$y_1 = x^i$$

$$y_2 = x^{-i}$$

the complementary function for the above differential equation is then

$$y = C_1.x^i + C_2.x^{-i}$$

in the above differential equation the method of systematic inspection could not find a direct solution but it could discover the behavior of the differential equation from which we derived the solution. We note here that when the dimension of the independent variable minus the dimension of the dependent variable is equal in any two terms in the differential equation then, if a seed is put opposite to any of these terms then the half of the inspected function created by this seed will be misleading by the process of calculate and balance, and you need to solve this half of the inspected function the same way I have done in the previous example. The following example gives more light on this problem.

2.4 example

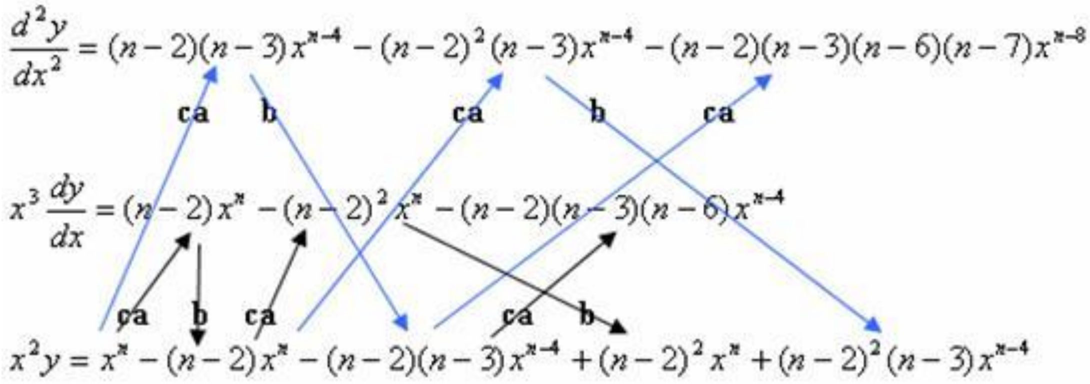
solve

$$\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + x^2y = 0$$ provided $-1 < x < +1$

here we note that the second and the third terms are having the same difference between the dimension of x and the dimension of y which is $3-1=2$ and $2-0=2$ respectively.

solution

for positive seed we put x^n opposite to x^2y . (the half of the inspected function created by this seed will be misleading).



$y_+ = x^{n-2} - (n-2)x^{n-2} - (n-2)(n-3)x^{n-6} + (n-2)^2x^{n-2} + (n-2)^2(n-3)x^{n-6}\ \dots$

for every calculation from x^2y to $\frac{x^3dy}{dx}$ divide by x^2 then differentiate once then multiply by x^3.

For every calculation from x^2y to $\frac{d^2y}{dx^2}$ divide by x^2 then differentiate twice.

We note here that if $n = 2$ then y_+ will be equal to x^0 or $y_+ = 1$, but for $n = 3$ the repetition problem of the power of x will appear so we use $y_+ = cx$ (because all other terms in the series of y_+ will be eliminated for $n = 3$) and we do the same as we have done in the previous example.

$y_+ = cx$

substituting in the differential equation we get

$$0 + x^3.c + x^2.cx = x^3$$

or

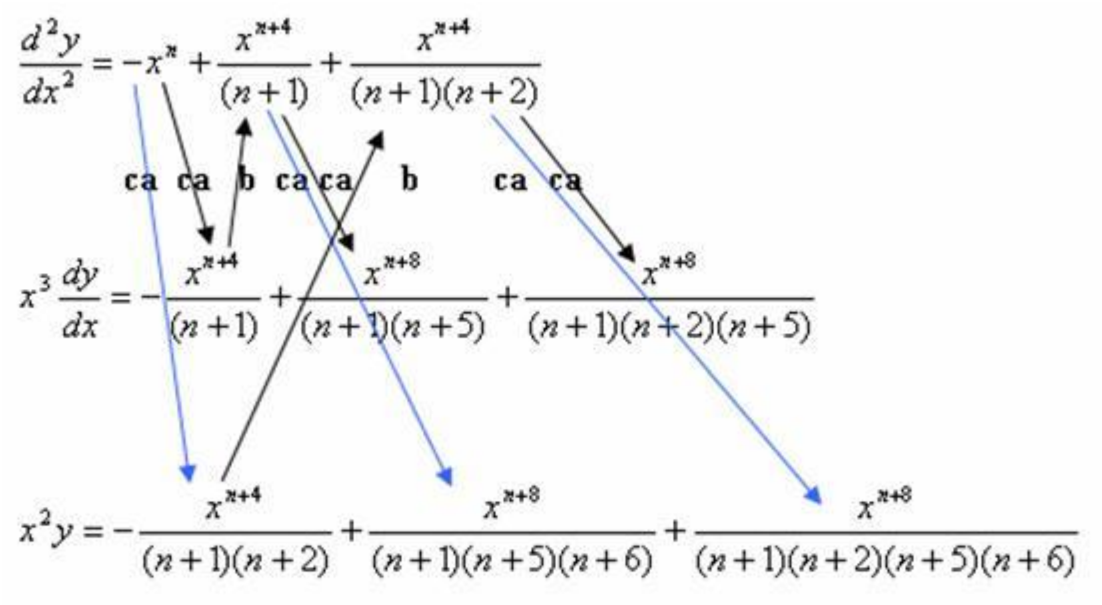
$$2cx^3 = x^3$$

then $c = \frac{1}{2}$

then for $n = 3$ $y_+ = \frac{1}{2}x$

now we go for the negative seed

put $-x^n$ opposite to $\frac{d^2y}{dx^2}$ (the half of the inspected function created by this seed will be correct).



$$y_- = -\frac{x^{n+2}}{(n+1)(n+2)} + \frac{x^{n+6}}{(n+1)(n+5)(n+6)} + \frac{x^{n+6}}{(n+1)(n+2)(n+5)(n+6)}$$

for every calculation from $\frac{d^2y}{dx^2}$ to $x^3\frac{dy}{dx}$ we integrate once then multiply by x^3.

for every calculation from $\frac{d^2y}{dx^2}$ to x^2y we integrate twice then multiply by x^2.

for $n = 2$

$$y_- = -\frac{x^4}{3\cdot 4} + \frac{x^8}{3\cdot 7\cdot 8} + \frac{x^8}{3\cdot 4\cdot 7\cdot 8} \ldots$$

for $n = 3$

$$y_- = -\frac{x^5}{4\cdot 5} + \frac{x^9}{4\cdot 8\cdot 9} + \frac{x^9}{4\cdot 5\cdot 8\cdot 9} \ldots$$

final solution

for $n = 2$

$$y = 1 - \frac{x^4}{3\cdot 4} + \frac{x^8}{3\cdot 7\cdot 8} + \frac{x^8}{3\cdot 4\cdot 7\cdot 8} \ldots$$

for $n = 3$

$$y = \frac{1}{2}x - \frac{x^5}{4\cdot 5} + \frac{x^9}{4\cdot 8\cdot 9} + \frac{x^9}{4\cdot 5\cdot 8\cdot 9} \ldots$$

3. Solving partial differential equations

Inspecting functions:

In the case of partial differential equations in terms of x and y let the starting seed to be $x^n y^m$.

3.1 example

solve the following differential equation (torsion problem)

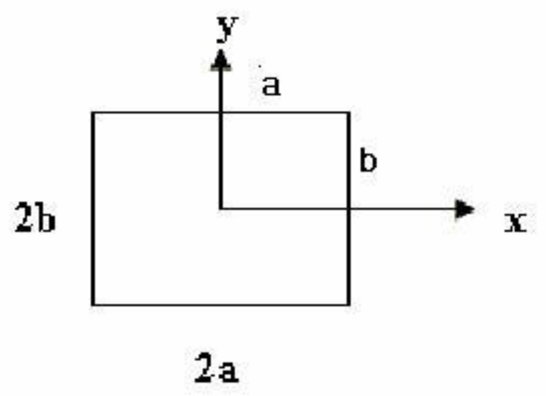
$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=-2G\theta$$

where G is the modulus of rigidity, θ is the angle of twist then $G\theta$ is a constant

for the following boundary conditions

$\Phi=0$ at $x=a$ & $x=-a$

$\Phi=0$ at $y=b$ & $y=-b$



Solution

Positive seed

Put $x^n y^m$ opposite to $\frac{\partial^2\Phi}{\partial x^2}$

$$\Phi_{+}=\frac{x^{n+2}y^{m}}{(n+1)(n+2)}-\frac{m(m-1)x^{n+4}y^{m-2}}{(n+1)(n+2)(n+3)(n+4)}+\frac{m(m-1)(m-2)(m-3)x^{n+6}y^{m-4}}{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}$$

$$\frac{\partial^2\Phi}{\partial x^2}=x^n y^m-\frac{m(m-1)x^{n+2}y^{m-2}}{(n+1)(n+2)}+\frac{m(m-1)(m-2)(m-3)x^{n+4}y^{m-4}}{(n+1)(n+2)(n+3)(n+4)}$$

ca b ca b

$$\frac{\partial^2\Phi}{\partial y^2}=\frac{m(m-1)x^{n+2}y^{m-2}}{(n+1)(n+2)}-\frac{m(m-1)(m-2)(m-3)x^{n+4}y^{m-4}}{(n+1)(n+2)(n+3)(n+4)}\cdots$$

in the case of calculation, integrate twice with respect to x then calculate $\frac{\partial^2\Phi}{\partial y^2}$ by differentiating twice with respect to y.

to calculate Φ integrate twice with respect to x

Φ_+ is convergent for $n \geq 0$, m any value

Negative seed

put $x^n y^m$ opposite to $\frac{\partial^2\Phi}{\partial y^2}$

$$\Phi_{-}=-\frac{x^{n}y^{m+2}}{(m+1)(m+2)}+\frac{n(n-1)x^{n-2}y^{m+4}}{(m+1)(m+2)(m+3)(m+4)}-\frac{n(n-1)(n-2)(n-3)x^{n-4}y^{m+6}}{(m+1)(m+2)(m+3)(m+4)(m+5)(m+6)}$$

$$\frac{\partial^2\Phi}{\partial x^2}=-\frac{n(n-1)x^{n-2}y^{m+2}}{(m+1)(m+2)}+\frac{n(n-1)(n-2)(n-3)x^{n-4}y^{m+4}}{(m+1)(m+2)(m+3)(m+4)}\cdots$$

ca b ca b

$$\frac{\partial^2\Phi}{\partial y^2}=-x^n y^m+\frac{n(n-1)x^{n-2}y^{m+2}}{(m+1)(m+2)}-\frac{n(n-1)(n-2)(n-3)x^{n-4}y^{m+4}}{(m+1)(m+2)(m+3)(m+4)}$$

in the case of calculation integrate twice with respect to y then calculate $\frac{\partial^2 \Phi}{\partial x^2}$ by differentiating twice with respect to x.

to calculate Φ integrate twice with respect to y

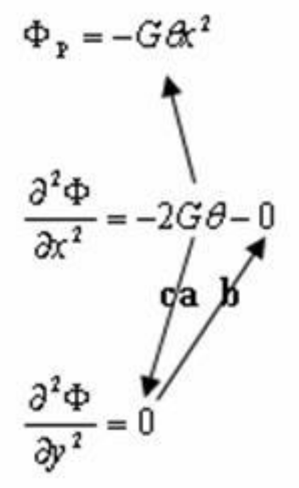
Φ_+ is convergent for $n \geq 0$, m any value

Φ_- is convergent for $m \geq 0$, n any value.

$\Phi = \Phi_+ + \Phi_-$

Radius of convergence is $m \geq 0$, $n \geq 0$

3.2 Particular integral



Then

$\Phi_p = -G\vartheta x^2$ (the other way around is also ok i.e. $\Phi_p = -G\vartheta y^2$)

now let us get enough functions by substituting different values of n and m into the above inspected function form :

$\Phi_{00} = 0.5x^2 – 0.5y^2$

$\Phi_{02} = 0.5x^2y^2 – 0.0833y^4 – 0.0833x^4$

$\Phi_{04} = 0.5x^2y^4 – 0.0333y^6 – 0.5x^4y^2 + 0.0333x^6$

$\Phi_{06} = 0.5x^2y^6 – 0.0179y^8 – 1.25x^4y^4 + 0.5x^6y^2 – 0.0179x^8$

$\Phi_{08} = 0.5x^2y^8 – 0.0111y^{10} – 2.333x^4y^6 + 2.333x^6y^4 – 0.5x^8y^2 + 0.0111x^{10}$

$\Phi_{10} = 0.5x^2y^{10} – 0.007575y^{12} – 3.75x^4y^8 + 7x^6y^6 – 3.75x^8y^4 + 0.5x^{10}y^2 - 0.007575x^{12}$

$\Phi_{012} = 0.5x^2y^{12} – 0.0054945y^{14} - 5.5x^4y^{10} + 16.5x^6y^8 – 16.5x^8y^6 + 5.5x^{10}y^4 – 0.5x^{12}y^2 + 0.0054945x^{14}$

here I have selected even values for n & m because of the symmetry of boundary conditions.

now multiplying any of these inspected functions with a constant will not affect it.

so to make them easier to handle multiply

Φ_{00} by 2
Φ_{02} by 12
Φ_{04} by 30
Φ_{06} by 56
Φ_{08} by 90
Φ_{10} by 132
Φ_{12} by 182

As a result the inspected functions become

$\Phi_{00} = x^2 - y^2$

$\Phi_{02} = 6x^2y^2 - y^4 - x^4$

$\Phi_{04} = 15x^2y^4 - y^6 - 15x^4y^2 + x^6$

$\Phi_{06} = 28x^2y^6 - y^8 - 70x^4y^4 + 28x^6y^2 - x^8$

$\Phi_{08} = 45x^2y^8 - y^{10} - 210x^4y^6 + 210x^6y^4 - 45x^8y^2 + x^{10}$

$\Phi_{010} = 66x^2y^{10} - y^{12} - 495x^4y^8 + 924x^6y^6 - 495x^8y^4 + 66x^{10}y^2 - x^{12}$

$\Phi_{012} = 91x^2y^{12} - y^{14} - 1001x^4y^{10} + 3003x^6y^8 - 3003x^8y^6 + 1001x^{10}y^4 - 91x^{12}y^2 + x^{14}$

Now

$\Phi = C + C_{00}\Phi_{00} + C_{02}\Phi_{02} + C_{04}\Phi_{04} + C_{06}\Phi_{06} + C_{08}\Phi_{08} + C_{010}\Phi_{010} + C_{012}\Phi_{012} + PS$

Or

$$
\begin{aligned}
\Phi &= C \\
&+C_{00}(x^2 - y^2) \\
&+C_{02}(6x^2y^2 - y^4 - x^4) \\
&+C_{04}(15x^2y^4 - y^6 - 15x^4y^2 + x^6) \\
&+C_{06}(28x^2y^6 - y^8 - 70x^4y^4 + 28x^6y^2 - x^8) \\
&+C_{08}(45x^2y^8 - y^{10} - 210x^4y^6 + 210x^6y^4 - 45x^8y^2 + x^{10}) \\
&+C_{010}(66x^2y^{10} - y^{12} - 495x^4y^8 + 924x^6y^6 - 495x^8y^4 + 66x^{10}y^2 - x^{12}) \\
&+C_{012}(91x^2y^{12} - y^{14} - 1001x^4y^{10} + 3003x^6y^8 - 3003x^8y^6 + 1001x^{10}y^4 - 91x^{12}y^2 + x^{14}) \\
&-G\vartheta x^2
\end{aligned}
$$

now subjecting the above formula to the boundary conditions we get,

$\Phi = 0$ at $x = a$ and $x = -a$

$0 = C$

$+ C_{00}(a^2 - y^2)$

$+ C_{02}(6a^2y^2 - y^4 - a^4)$

$+ C_{04}(15a^2y^4 - y^6 - 15a^4y^2 + a^6)$

$+ C_{06}(28a^2y^6 - y^8 - 70a^4y^4 + 28a^6y^2 - a^8)$

$+ C_{08}(45a^2y^8 - y^{10} - 210a^4y^6 + 210a^6y^4 - 45a^8y^2 + a^{10})$

$+ C_{010}(66a^2y^{10} - y^{12} - 495a^4y^8 + 924a^6y^6 - 495a^8y^4 + 66a^{10}y^2 - a^{12})$

$+ C_{012}(91a^2y^{12} - y^{14} - 1001a^4y^{10} + 3003a^6y^8 - 3003a^8y^6 + 1001a^{10}y^4 - 91a^{12}y^2 + a^{14})$

$-G\vartheta a^2$

now collecting the constants accompanied with y^0, y^2, y^4, y^6 we get the following equations :

$C + a^2C_{00} - a^4C_{02} + a^6C_{04} - a^8C_{06} + a^{10}C_{08} - a^{12}C_{010} + a^{14}C_{012} - G\theta a^2 = 0$ ..(1)

$-C_{00} + 6a^2C_{02} - 15a^4C_{04} + 28a^6C_{06} - 45a^8C_{08} + 66a^{10}C_{010} - 91a^{12}C_{012} = 0$ ..(2)

$-C_{02} + 15a^2C_{04} - 70a^4C_{06} + 210a^6C_{08} - 495a^8C_{010} + 1001a^{10}C_{012} = 0$ ..(3)

$-C_{04} + 28a^2C_{06} - 210a^4C_{08} + 924a^6C_{010} - 3003a^8C_{012} = 0$ ..(4)

Subjecting the Φ formula to the second boundary condition,

Φ = 0 at *y = b* and *y = -b*

$0 = C$
$+ C_{00}(x^2 - b^2)$
$+ C_{02}(6x^2b^2 - b^4 - x^4)$
$+ C_{04}(15x^2b^4 - b^6 - 15x^4b^2 + x^6)$
$+ C_{06}(28x^2b^6 - b^8 - 70x^4b^2 + 28x^6b^2 - x^8)$
$+ C_{08}(45x^2b^8 - b^{10} - 210x^4b^6 + 210x^6b^4 - 45x^8b^2 + x^{10})$
$+ C_{010}(66x^2b^{10} - b^{12} - 495x^4b^8 + 924x^6b^6 - 495x^8b^4 + 66x^{10}b^2 - x^{12})$
$+ C_{012}(91x^2b^{12} - b^{14} - 1001x^4b^{10} + 3003x^6b^8 - 3003x^8b^6 + 1001x^{10}b^4 - 91x^{12}b^2 + x^{14})$
$-G\vartheta x^2$

now collecting the constants accompanied with x^0, x^2, x^4, x^6 we get the following equations

$C - b^2C_{00} - b^4C_{02} - b^6C_{04} - b^8C_{06} - b^{10}C_{08} - b^{12}C_{010} - b^{14}C_{012} = 0$ ..(5)

$C_{00} + 6b^2C_{02} + 15b^4C_{04} + 28b^6C_{06} + 45b^8C_{08} + 66b^{10}C_{010} + 91b^{12}C_{012} - G\theta = 0$ ..(6)

$-C_{02} - 15b^2C_{04} - 70b^4C_{06} - 210b^6C_{08} - 495b^8C_{010} - 1001b^{10}C_{012} = 0$ ..(7)

$C_{04} + 28b^2C_{06} + 210b^4C_{08} + 924b^6C_{010} + 3003b^8C_{012} = 0$ ..(8)

Now solve these equations simultaneously to get the unknown constants C, C_{00},

C_{02}, C_{04}, C_{06}, C_{08}, C_{010}, C_{012}.
In the above example I created 8 equations, but you can create more if you need more accuracy.

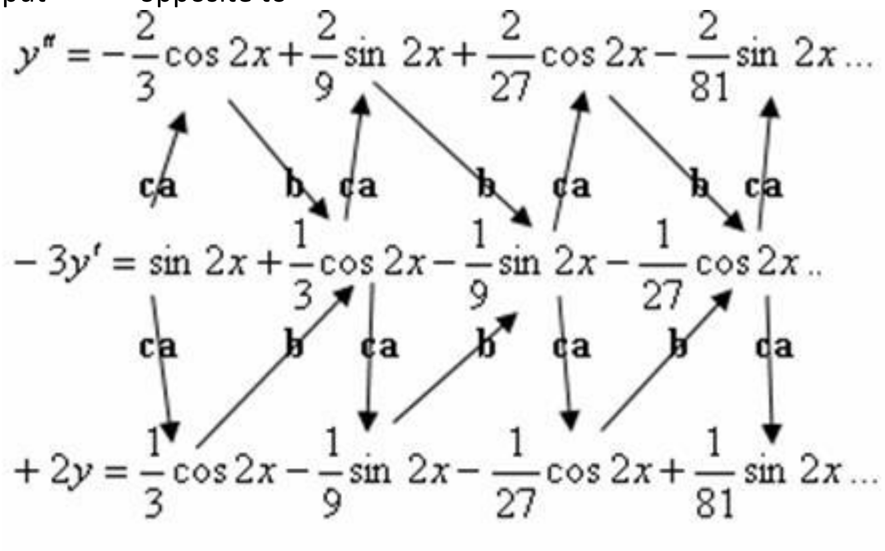
4. Solving differential equations that include functions

Find the particular integral for

$y'' - 3y' + 2y = \sin 2x$

solution

put $\sin 2x$ opposite to $-3y$



knowing that :

$$\int \sin 2x dx = -\frac{1}{2}\cos 2x$$

$$\int \cos 2x dx = +\frac{1}{2}\sin 2x$$

$$\frac{d \sin 2x}{dx} = 2\cos 2x$$

$$\frac{d \cos 2x}{dx} = -2\sin 2x$$

from this we find that :

$y_p = -0.05\sin 2x + 0.15\cos 2x$

in the case of calculation from $-3y'$ to y'' divide by -3 then integrate once w.r.t. x

in the case of calculation from $-3y'$ to $+2y$ divide by –3 then differentiate once w.r.t. x

5. Solving simple non-linear differential equations

Solving a non-linear differential equation is like solving a polynomial, so at the beginning we will solve a polynomial. This method solves simple non-linear differential equations. solving non-linear differential equations using this method requires more research. Let's start by solving the following quadratic equation,

5.1 solve

$x^2 + x = 5$

put the equation in two rows

$x^2 = 5$

$x =$

$sum = x^2 + x = 5$

make a calculation

$$x^2 = 5$$
$$\downarrow$$
$$x = \sqrt{5}$$

$sum = 7.2361$

now subtract 5 from the sum

$$x^2 = 5$$
$$\downarrow$$
$$x = \sqrt{5}$$

$sum = 7.2361$

-5

$result = 2.2361$

now make a balance

$x^2 = 5 - 2.2361 = 2.7639$

$x =$

now make the calculation by neglecting what you had previously for x

$$x^2 = 2.7639$$
$$\sqrt{\ }\downarrow$$
$$x = 1.6625$$

$sum = 4.4264$

-5

$result = -0.5736$

make a balance

$$x^2 = 2.7639 + 0.5736 = 3.3375$$

$$\swarrow \sqrt{\ }$$

$$x = 1.82688$$

calculate by neglecting the previous value of x

$sum = 5.16438$

-5

$result = 0.16438$

proceed for more accuracy until you get the difference (result) = 0, this will give us,

$x = 1.79129$

now we need to get the other root.

If you try to put 5 opposite to x as follows, you will get a divergent solution as you see down :

$$x = 5$$

$$\downarrow$$

$$x^2 = 25$$

$sum = 30$

-5

$result = 25$

balance

$$x = 5 - 25 = -20$$

$$\swarrow$$

$$x^2 = 400$$

$sum = -380$

-5

$result = -385$

the result is that x is diverging as we proceed. So how to get the other root ?

our equation is

$$x^2 + x = 5$$

dividing the equation by x will not affect it, so we get

$$x + 1 = \frac{5}{x}$$

or

$$x - \frac{5}{x} = -1$$

now you can start the process of calculate and balance as follows,

$$x = -1$$

$$-\frac{5}{x} =$$

$$sum = x - \frac{5}{x} = -1$$

now we make a calculation

$$x = -1$$

$$-\frac{5}{x} = 5$$

$sum = 4$ subtract -1(or add 1)

$+1$

$result = 5$

make a balance

$$x = -1 - 5 = -6$$

$$-\frac{5}{x} = 0.833$$

$sum = -5.167$ subtract -1(or add 1)

$+1$

$result = -4.167$

make a balance

$$x = -6 + 4.167 = -1.83$$

$$-\frac{5}{x} = 2.727$$

$sum = 0.894$ subtract -1(or add 1)

$+1$

$result = 1.894$

if you proceed with the process of calculate and balance you will get :

$x = -2.791$, which is the second root.
This method works fine for real roots, but for imaginary roots it gives a series

that is a function of i ($i = \sqrt{-1}$)

5.2 Solve

$$(\frac{dy}{dx})^2 + y = x$$

Solution

Calculate

$$y = x$$

$$(\frac{dy}{dx})^2 = 1$$

$sum = x + 1$

$-x$

result 1

balance

$$y = x - 1$$

$$\downarrow$$

$$(\frac{dy}{dx})^2 = 1$$

$sum = x$

$-x$

result 0

so the first root is

$y = x - 1$

to find the second root try the following

calculate

$$(\frac{dy}{dx})^2 = x$$

$$\downarrow$$

$$y = \frac{2}{3}x^{\frac{3}{2}}$$

$sum = x + \frac{2}{3}x^{\frac{3}{2}}$

$-x$

result $\frac{2}{3}x^{\frac{3}{2}}$

Balance

$$(\frac{dy}{dx})^2 = x - \frac{2}{3}x^{\frac{3}{2}}$$

$$\downarrow$$

$$y = \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx$$

$sum = x - \frac{2}{3}x^{\frac{3}{2}} + \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx$

$-x$

$$result = -\frac{2}{3}x^{\frac{3}{2}} + \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx$$

Balance

$$\left(\frac{dy}{dx}\right)^2 = x + \frac{2}{3}x^{\frac{3}{2}} - \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx$$

$$y = \int (x + \frac{2}{3}x^{\frac{3}{2}} - \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx)^{\frac{1}{2}} dx$$

$$sum = x + \frac{2}{3}x^{\frac{3}{2}} - \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx + \int (x + \frac{2}{3}x^{\frac{3}{2}} - \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx)^{\frac{1}{2}} dx$$

$-x$

proceed to get a more convergent solution, so

$$y = \cdots \int (x + \frac{2}{3}x^{\frac{3}{2}} - \int (x - \frac{2}{3}x^{\frac{3}{2}})^{\frac{1}{2}} dx)^{\frac{1}{2}} dx$$

I did not check the convergence of this second root. But if the solution is divergent or is identical to the first root, then you need to change the structure of the differential equation to get the second root as I have done with the quadratic equation in section 5.1. checking the convergence of this root is hard. the whole subject of non-linear differential equations requires extensive research.

6. Solving simultaneous ordinary linear differential equations

Solve the following set of ordinary differential equations :

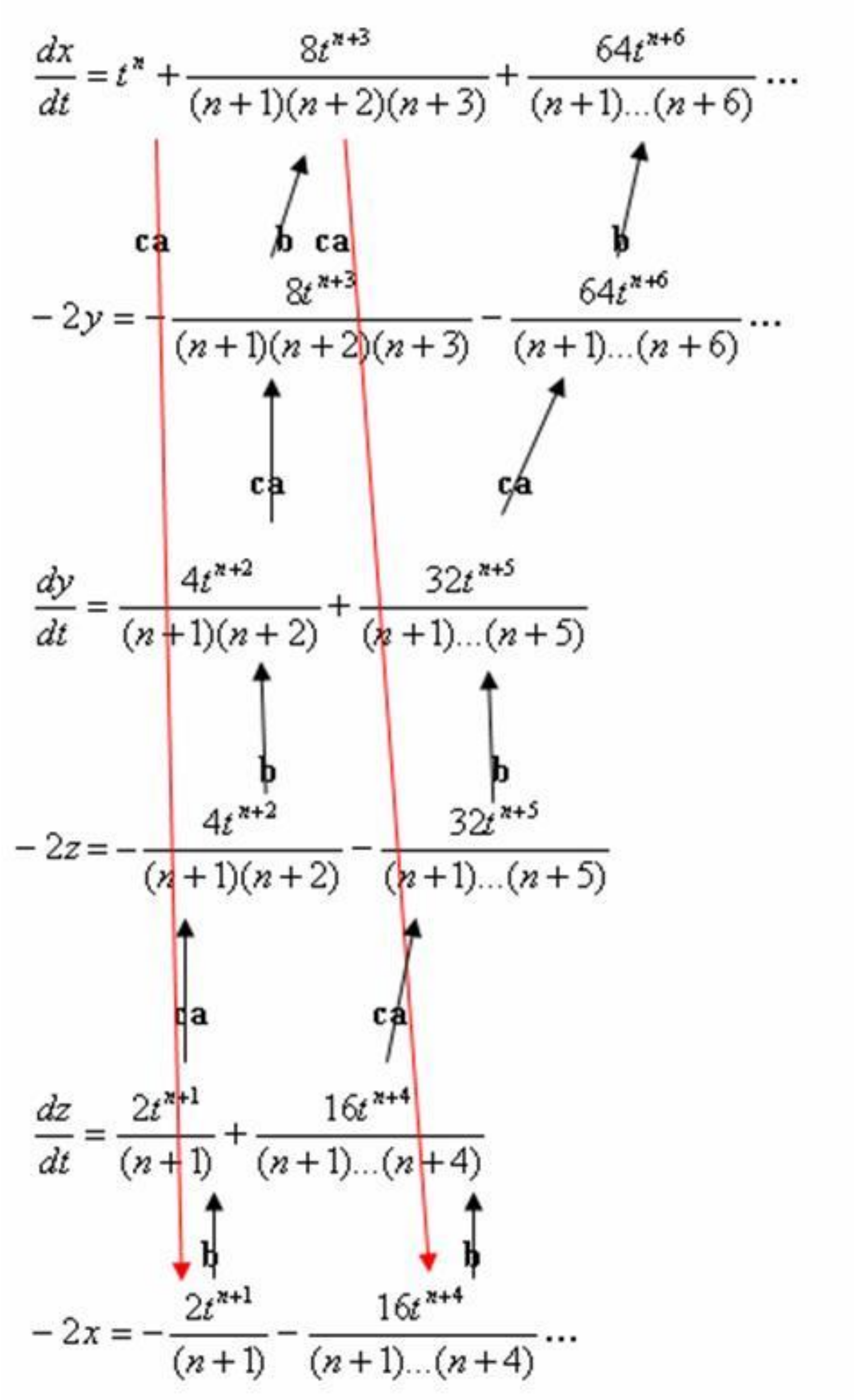
$$\frac{dx}{dt} - 2y = 0 \quad \text{.... (1)}$$

$$\frac{dy}{dt} - 2z = 0 \quad \text{.... (2)}$$

$$\frac{dz}{dt} - 2x = 0 \quad \text{.... (3)}$$

Solution

Positive seed: put t^n opposite to $\frac{dx}{dt}$



make a calculation from $\frac{dx}{dt}$ in the first equation to $-2x$ in the third equation by integrating once then multiplying by -2. make a balance to $\frac{dz}{dt}$ then make a calculation to $-2z$ in the middle equation by integrating once and multiplying by -2. make a balance from $-2z$ to $\frac{dy}{dt}$ in the middle equation then make a calculation from $\frac{dy}{dt}$ in the middle equation to $-2y$ in the first equation by

integrating once and multiplying by -2 then make a balance from $-2y$ to $\frac{dx}{dt}$ in the first equation. Repeat the process till you get a satisfactory number of terms.
this gives

$$y_+ = \frac{4t^{n+3}}{(n+1)\ldots(n+3)} + \frac{32t^{n+6}}{(n+1)\ldots(n+6)} + \frac{256t^{n+9}}{(n+1)\ldots(n+9)} \ldots$$

$$z_+ = \frac{2t^{n+2}}{(n+1)(n+2)} + \frac{16t^{n+5}}{(n+1)\ldots(n+5)} + \frac{128t^{n+8}}{(n+1)\ldots(n+8)} \ldots$$

$$x_+ = \frac{t^{n+1}}{(n+1)} + \frac{8t^{n+4}}{(n+1)\ldots(n+4)} + \frac{64t^{n+7}}{(n+1)\ldots(n+7)} \ldots$$

negative seed

put $-t^n$ opposite to $-2y$

$$\frac{dx}{dt} = \frac{1}{8}n(n-1)(n-2)t^{n-3} + \frac{1}{64}n(n-1)\ldots(n-5)t^{n-6} \ldots$$

ca b ca b

$$-2y = -t^n - \frac{1}{8}n(n-1)(n-2)t^{n-3} - \frac{1}{64}n(n-1)\ldots(n-5)t^{n-6} \ldots$$

ca ca

$$\frac{dy}{dt} = \frac{1}{2}n.t^{n-1} + \frac{1}{16}n(n-1)\ldots(n-3)t^{n-4} \ldots$$

b b

$$-2z = -\frac{1}{2}n.t^{n-1} - \frac{1}{16}n(n-1)\ldots(n-3)t^{n-4} \ldots$$

ca ca

$$\frac{dz}{dt} = \frac{1}{4}n(n-1)t^{n-2} + \frac{1}{32}n(n-1)\ldots(n-4)t^{n-5} \ldots$$

b b

$$-2x = -\frac{1}{4}n(n-1)t^{n-2} - \frac{1}{32}n(n-1)\ldots(n-4)t^{n-5} \ldots$$

make a calculation from $-2y$ in the first equation to $\frac{dy}{dt}$ in the middle equation by dividing by -2 then differentiating once then make a balance in the middle equation form $\frac{dy}{dt}$ to $-2z$. make a calculation from $-2z$ in the middle equation to $\frac{dz}{dt}$ in the third equation by dividing by -2 then differentiating once. Make a balance in the third equation from $\frac{dz}{dt}$ to $-2x$ in the same equation then make a calculation from $-2x$ in the third equation to $\frac{dx}{dt}$ in the first equation by dividing by -2 and differentiating once. Make a balance in the first equation from $\frac{dx}{dt}$ to $-2y$ in the same equation then repeat the process mentioned above.

this gives

$$y_- = \frac{1}{2}t^n + \frac{1}{16}n(n-1)(n-2)t^{n-3} + \frac{1}{128}n(n-1)\ldots(n-5)t^{n-6} \ldots$$

$$z_- = \frac{1}{4}nt^{n-1} + \frac{1}{32}n(n-1)\ldots(n-3)t^{n-4} + \frac{1}{256}n(n-1)\ldots(n-6)t^{n-7}$$

$$x_- = \frac{1}{8}n(n-1)t^{n-2} + \frac{1}{64}n(n-1)\ldots(n-4)t^{n-5} + \frac{1}{512}n(n-1)\ldots(n-7)t^{n-8}$$

For $n=0$

$$y = C_1\left(\frac{1}{2} + \frac{4t^3}{3!} + \frac{32t^6}{6!} + \frac{256t^9}{9!}\ldots\right)$$

$$z = C_1\left(\frac{2t^2}{2!} + \frac{16t^5}{5!} + \frac{128t^8}{8!}\ldots\right)$$

$$x = C_1\left(t + \frac{8t^4}{4!} + \frac{64t^7}{7!}\ldots\right)$$

for $n=1$

$$y = C_2\left(\frac{t}{2} + \frac{4t^4}{4!} + \frac{32t^7}{7!} + \frac{256t^{10}}{10!}\ldots\right)$$

$$z = C_2\left(\frac{1}{4} + \frac{2t^3}{3!} + \frac{16t^6}{6!} + \frac{128t^9}{9!}\ldots\right)$$

$$x = C_2\left(\frac{t^2}{2!} + \frac{8t^5}{5!} + \frac{64t^8}{8!}\ldots\right)$$

for $n=2$

$$y = C_3\left(\frac{t^2}{2} + \frac{8t^5}{5!} + \frac{64t^8}{8!} + \frac{512t^{11}}{11!}\ldots\right)$$

$$z = C_3\left(\frac{t}{2} + \frac{4t^4}{4!} + \frac{32t^7}{7!} + \frac{256t^{10}}{10!}\ldots\right)$$

$$x = C_3\left(\frac{1}{4} + \frac{2t^3}{3!} + \frac{16t^6}{6!} + \frac{128t^9}{9!}\ldots\right)$$

if you try $n > 2$ you will get similar functions .

final solution

$$y = C_1(\frac{1}{2} + \frac{4t^3}{3!} + \frac{32t^6}{6!} + \frac{256t^9}{9!}...)$$
$$+ C_2(\frac{t}{2} + \frac{4t^4}{4!} + \frac{32t^7}{7!} + \frac{256t^{10}}{10!}...)$$
$$+ C_3(\frac{t^2}{2} + \frac{8t^5}{5!} + \frac{64t^8}{8!} + \frac{512t^{11}}{11!}...)$$
$$z = C_1(\frac{2t^2}{2!} + \frac{16t^5}{5!} + \frac{128t^8}{8!}...)$$
$$+ C_2(\frac{1}{4} + \frac{2t^3}{3!} + \frac{16t^6}{6!} + \frac{128t^9}{9!}...)$$
$$+ C_3(\frac{t}{2} + \frac{4t^4}{4!} + \frac{32t^7}{7!} + \frac{256t^{10}}{10!}...)$$
$$x = C_1(t + \frac{8t^4}{4!} + \frac{64t^7}{7!}...)$$
$$+ C_2(\frac{t^2}{2!} + \frac{8t^5}{5!} + \frac{64t^8}{8!}...)$$
$$+ C_3(\frac{1}{4} + \frac{2t^3}{3!} + \frac{16t^6}{6!} + \frac{128t^9}{9!}...)$$

in the previous example we had two terms per equation so the process was very simple. But if we have more than two terms in one or some of the simultaneous differential equations then the process will be more difficult. The following example shows this and shows also how to deal with simultaneous partial differential equations.

7. solving simultaneous linear partial differential equations

Find one variation of the particular integral for the following set of linear partial differential equations:

$$u + v + w = x^2$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = y^2$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = z^2$$

the method of solution is as follows

1 – put x^2 opposite to u , y^2 opposite to $\frac{\partial v}{\partial y}$, and z^2 opposite to $\frac{\partial^2 w}{\partial z^2}$

2 – calculate $\frac{\partial u}{\partial x}$ from u and $\frac{\partial^2 u}{\partial x^2}$ from u

3 – calculate v from $\frac{\partial v}{\partial y}$ and $\frac{\partial^2 v}{\partial y^2}$ from $\frac{\partial v}{\partial y}$

4 - calculate w from $\frac{\partial^2 w}{\partial z^2}$ and $\frac{\partial w}{\partial z}$ from $\frac{\partial^2 w}{\partial z^2}$

5a – in the second equation balance 2x from ∂u/∂x into ∂v/∂y and balance $z^3/3$ from ∂w/∂z into ∂u/∂x. (not necessarily into ∂u/∂x, you can balance ∂w/∂z into ∂v/∂y).

5b – in the third equation balance 2 from $\partial^2u/\partial y^2$ into $\partial^2v/\partial y^2$ and balance 2y from $\partial^2v/\partial y^2$ into $\partial^2u/\partial x^2$.

6 – now make a calculation from the balances mentioned in (5a)&(5b) as follows,

6a – in the second equation make a calculation form –2x of ∂v/∂y into v in the first equation, this gives –2xy and make a calculation from $-z^3/3$ of ∂u/∂x into u in the first equation, this gives $-z^3x/3$

6b – in the third equation, make a calculation from –2 of $\partial^2v/\partial y^2$ into ∂v/∂y in the second equation, this gives –2y, and make a calculation from –2 of $\partial^2v/\partial y^2$ into v in the first equation, this gives $-2y^2$.

7 – now make the first balance in the first equation by balancing v into u. that is to balance $(y^3/3 - 2xy - y^2)$ from v into u, and balance u and w into v, that is to balance $(yx^2 - z^3x)$ from u and $z^4/12$ from w into v. by finishing this process we finish the first run of calculate and balance process.

Next we will start the second run.

8a –make a calculation from u into ∂u/∂x, this gives 2y in ∂u/∂x, and make a calculation from u into $\partial^2u/\partial x^2$, this gives 0.

8b – make a calculation from v into ∂v/∂y, this gives x^2 in ∂v/∂y, and make a calculation from v into $\partial^2v/\partial y^2$, this gives 0.

9 - now make balances in every equation as follows,

9a – in the second equation, we need to balance 2y form ∂u/∂x into ∂v/∂y, but since the second equation is already balanced, that is –2y already exists, so the balance is not necessary.

(note) : after every calculation run for an equation, we should check the balance of the equation so that no disturbance in the balance occurs.

9b – in the third equation, no balance required, since the balance results are zeros.

10 – in the second equation, make a calculation for $-x^2$ from ∂u/∂x into u, this gives -$x^3/3$, and make a calculation for $-x^2$ from ∂u/∂x into $\partial^2u/\partial x^2$, this gives –2x.

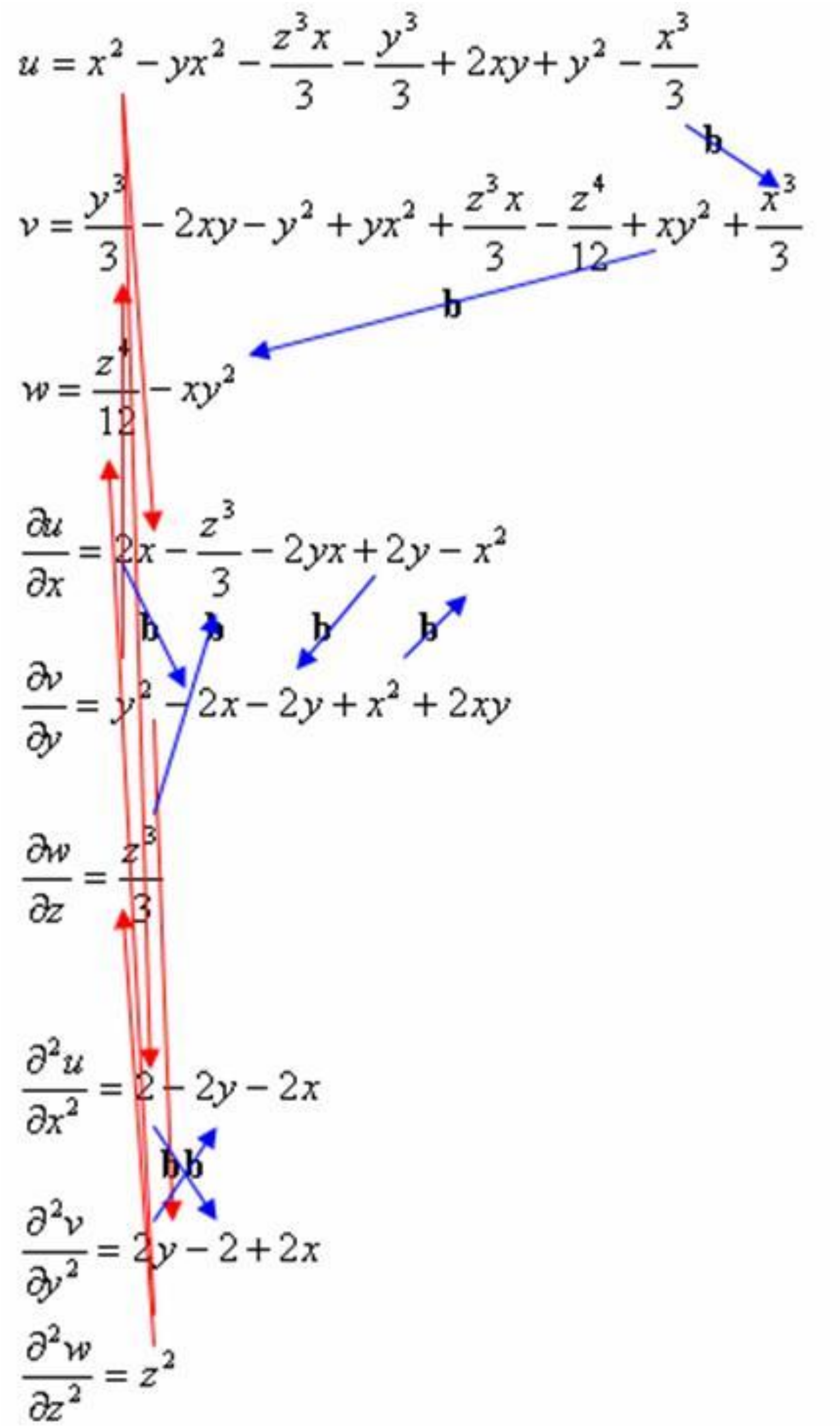
11 - in the third equation make a balance for –2x from $\partial^2u/\partial x^2$ into $\partial^2v/\partial y^2$.

12 – in the third equation make a calculation for +2x from $\partial^2v/\partial y^2$ into ∂v/∂y in the second equation, this gives +2xy, and make a calculation for +2x from $\partial^2v/\partial y^2$ into v in the first equation, this gives xy^2.

13 – now make the second run balance in the first equation, that is to balance from u into v which balances $-x^3/3$ from u into v, and make a balance from v into z which balances xy^2 from v into z.

by this we terminate the process of calculation, because any calculation process gives zero in the second and the third equation. Note that since the number of arrows pointing from one term to the other is much, so I did not put all the arrows.

Solution



Conclusion

I want to conclude this work by stating that this method gives us a new way of thinking of differential equations. that's why I used simple ones. Just to clear up the concept. Every differential equation has its own behavior and should be treated and analyzed differently. Differential equations with more than two terms need computer to be solved, since the number of the processes of calculate and balance becomes huge.