RELATIONSHIP BETWEEN THE CLASSICAL NUMERICAL RANGE AND ESSENTIAL NUMERICAL RANGES

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Abstract

In this paper we discuss the relationships between the numerical range and the essential numerical range. We prove two theorems in this paper that show how the numerical range and the essential numerical range are related. Finally, we discuss the roles of the essential numerical range in operator theory.

Keywords: Numerical range, Essential Numerical range, Orthonormal Basis and Inner Product.

1 Introduction

Let H be a Hilbert space equipped with the inner product $\langle .,. \rangle$ and let $B(H)$ be the algebra of bounded linear operators acting on H. The relationship between the numerical range and the essential numerical range is given in the result by John Lancaster which is also reinforced by J. Christophe's theorem.

2 Basic concepts and preliminaries

Here we start by defining some key terms that are useful in the sequel.

Definition 2.1. Numerical range $W(T)$ of $T \in B(H)$ is the collection of all complex numbers of the form $\langle Tx, x \rangle$ where x is a unit vector in H i.e

$$W(T) = \{\langle Tx, x \rangle : x \in H, \|x\| = 1\}.$$

Definition 2.2. An inner product on a vector space V is a map $\langle .,. \rangle: V \times V \to \mathbb{K}$ such that $\forall\ x, y\ z\ \in V$ and $\lambda \in \mathbb{K}$, the following properties are satisfied:

(i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$,if and only if $x = 0$.

(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.

(iii) $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$.

(iv) $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

The ordered pair $(V, \langle ., .\rangle)$ is called an inner product.

Definition 2.3. Orthonormal basis; an orthonormal set E is a basis for H if every $x \in H$ can be written uniquely in the form

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$

For some $\alpha_k \in \mathbb{K}$ and $e_k \in E$.

3 Main results

Theorem 3.1 (John Lancaster theorem)**.** For $T \in H$ we have

$\overline{W(T)} = conv\{W(T) \cup W_e(T)\}$-(see [21])

Proof. Clearly, $W(\alpha T + \beta) = \alpha W(T) + \beta$ for all $\alpha, \beta \in C$. (Therefore by rotation and translation, we can assume that $\overline{W(T)}$ is contained in the closed right half plane and

$0 \in Ext\left(\overline{W(T)}\right) - W(T)$. Then there exists a sequence$\{x \quad\}_{n=1}^{\infty}$

of unit vectors of H such that $\langle Tx, x\rangle \to 0$. By weak sequential compactness of the unit ball of H, we can assume that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $x \in H$ with $\|x\| \leq 1$. We prove that x is the 0 vector, and hence

$0 \in W_e(T)$.

If $\|x\| = 1$, then $x_n \to x$ strongly. But;

$$|\langle Tx, x\rangle| \leq |\langle T(x - x_n), x\rangle| + |\langle Tx_n, x - x_n\rangle| + |\langle Tx_n, x_n\rangle|$$

$$\leq |\langle x - x_n, T^* x\rangle| + \|T\|\|x - x_n\| + |\langle Tx_n, x_n\rangle| \to 0$$

Hence $\langle Tx, x\rangle = 0$ and $0 \in W(T)$. So assume$0 < \|x\| < 1$. Clearly the operator ReT is positive since $W(T)$ is contained in the closed right half plane. Then;

$$\left\|(ReT)^{\frac{1}{2}}\right\|^2 = \langle (ReT)x_n, x_n\rangle$$

$$= Re\langle Tx_n, x_n\rangle \to 0,$$

so $\|(ReT)x_n\| \to 0$. This clearly yields $Re\langle Tx, x\rangle = 0$ so $\langle Tx, x\rangle$ is purely imaginary. On the other hand;

$$\langle T(x - x_n), x - x_n\rangle = \langle Tx, x - x_n\rangle - \langle Tx_n, x\rangle + \langle Tx_n, x_n\rangle \to \langle Tx, x\rangle$$

and

$\|x - x_n\|^2 = 1 - 2Re\langle x - x_n, x\rangle - \|x\|^2,$

so $\langle Ty_n, y_n\rangle \to -\langle Tx, x\rangle/(1 - \|x\|^2)$ where $y_n = \frac{(x-x_n)}{\|x-x_n\|}$. Thus we have produced a non-zero purely imaginary points in $W(T)$ which lie in the upper and lower half planes. However this implies that 0 is a non-extreme point of $W(T)$, thus completing the proof of the inclusion. The equality follows from the inclusion by the Krein-Milman theorem.

Theorem 3.2. Let T be an operator, then:

(i). *If $W_e(T) \subset W(T)$ then $W(T)$ is closed.*

(ii). *There exist normal finite rank operators R of arbitrarily small norm such that $W(T + R)$ is closed*

Proof. Assertion (i) is due to Theorem 4.3.1. We prove the second assertion and implicitly prove Lancaster's result.

We may find an orthonormal system $\{x_n\}$ such that the closure of the sequence $\{\langle Tx_n, x_n\rangle\}$ contains the boundary of the essential numerical range, $\delta W_e(T)$. Fix $\varepsilon > 0$. It is possible to find an integer p and scalars $z_j, 1 < j < p,$ with $|z| < \varepsilon$ such that;

$co\{\langle x_j, Tx_j\rangle + z_j : 1 < j < p\} \supset \delta W_e(T)$.

Thus, the finite rank operators,

$R = \sum_{1<J<P} z_j \otimes x_j$ has the property that $W(T+R)$ contains $W_e(T)$. We need this operator R. Indeed, setting $X = T + R$, we also have $W(X) \supset W_e(X)$. We then claim that $W(X)$ is closed (this claim implies assertion

(i)). By the contrary, there would exist, $z \in \delta\overline{W(X)}$.

Furthermore, since $W(X)$ is the convex hull of its extreme points, we could assume that such a z is an extreme point of $W(X)$. By suitable rotation and translation, we could assume that $0z = 0$ and that the imaginary axis is a line of support of $W(X)$. The projection property for $W(\cdot)$ would imply that $W(ReX) = (x, 0]$ for a certain negative number x, so that $0 \in W_e(X)$. Thus we would deduce from the projection property for $W_e(\cdot)$ that $o \in W_e(X)$; a contradiction. (see, [19])

4 Role of the essential numerical range

An operator $T \in H$ has the *small entry property* if for every $\varepsilon > 0$, there is a basis $\{x_n\}$ such that $|\langle Te_n, e_m\rangle| < \varepsilon$ for all n and m. The condition $0 \in W_e(T)$ is equivalent to the fact that the operator T has the *small entry property*. That is; if the operator T has the small entry property, then for any $\varepsilon > 0$, there is a basis so that all entries of the matrix of T have absolute value less than ε. In particular, the diagonal entries of the matrix must have an accumulation point λ with $|\lambda| < \varepsilon$ and since $W_e(T) = \{\lambda$: *there is an orthonormal sequence* $\{x_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty}\langle Tx_n, x_n\rangle = \lambda$, it's evident that $\lambda \in W_e(T)$. Now since $W_e(T)$ is closed, $0 \in W_e(T)$. We conclude that $0 \in W_e(T)$ is equivalent to the property that the operator T has the *small entry property*. Thus we infer that the essential numerical range serves to identify the class of operators that satisfy the *small entry property*. (see,[9])

We give the theorem by Q. F. Stout [12] that reinforces that the condition

$0 \in W_e(T)$ is equivalent to the fact that the operator T has the *small entry property*.

Theorem 4.1. *For any $T \in H$, the following conditions are equivalent:*

(a). $0 \in W_e(T)$

(b). *There is a basis ξ such that $T \in kernel(hull(\kappa_\xi))$.*

(c). *T has the small entry property.*

(d). *There exists a sequence of bases $\xi_{(n)}$ such that $T_{\xi(n)} \to 0$ uniformly in H. (see, [12])*

Proof. The proof can be found in [12, Theorem 2.3]

4.2 Zero diagonal operators

An operator $T \in H$ is called *zero diagonal* if there exists an orthonormal basis $\{x_n\}$ for H such that $\langle Te_n, e_n \rangle = 0$ for all n. We state the theorem below by D. Bakic [9] without proof.

Theorem 4.2. *Let $T \in H$ be a bounded operator on a separable Hilbert space, H. Then there exists an orthonormal basis $\{e_n\}$ for H such that $\lim\langle Te_n, e_n \rangle = 0$ if and only if 0 is in the essential numerical range of T.*

Thus from this theorem, we infer that 0 is in the essential numerical range. We conclude that the notion that an operator T is zero diagonal is equivalent to the fact that $0 \in W_e(T)$. Thus the essential numerical range also serves to identify zero diagonal operators (see, [1-9, 11-24]).

The essential numerical range plays an important role in solving problems from the operator theory. The list below of mutually equivalent conditions indicates the importance of the essential numerical range.

Theorem 4.3. *For an operator $A \in H$ the following conditions are mutually equivalent.*

(a). *There exists an orthonormal basis $\{e_n\}$ for H such that* $\lim_n \langle Ae_n, e_n\rangle = 0$.

(b). $0 \in W_e(A)$.

(c). *There exists an orthonormal sequence $\{a_n\}$ in H such that* $\lim_n \langle Aa_n, a_n\rangle = 0$.

(d). *There exists a sequence of unit vectors (x_n) in H weakly converging to* 0 *such that* $\lim_n Tx_n = 0$.

(e). *There exists an orthogonal projection $P \in H$ with an infinite dimensional range such that PAP is a compact operator.*

(f). *For each $\varepsilon > 0$ there exists an orthonormal basis $\{e_n\}$ for H such that $|\langle Ae_n, e_m\rangle| < \varepsilon$, for all n and m.*

(g). *For each $\varepsilon > 0$ and $p > 1$ there exists an orthonormal basis $\{e_n\}$ for H such that* $\sum_{n=1}^{\infty} |\langle Ae_n, e_n\rangle|^p < \varepsilon$.

(h). *There exists a sequence of zero diagonal operators A_n in H such that* $A = (norm) \lim_n A_n$.

(i). *There exists a zero diagonal operator $T \in H$ and a compact operator $K \in \mathcal{K}(H)$ such that $A = T + K$.*

(j). *There exists an operator $B \in H$ such that $A = B^*B - BB^*$. In this case A is self-adjoint necessarily.*

(k). *The spectrum of A has at least one non-negative limit point and at least one non-positive limit point. (see, [9])*

Proof. (a) ⇔ (b), This is due to the assertion of *Theorem 4.3.2* above.

(b), (c), (d) and (e) are equivalent, (see, [5]).

$(e) \Leftrightarrow (f) \Leftrightarrow (g)$

$(h) \Leftrightarrow (b)$

$(i) \Leftrightarrow (a)$

$(a) \Rightarrow (i)$: Let us take the orthonormal basis from (a) and define $K \in H$ by

$Ke_n = \langle Ae_n, e_n \rangle e_n$ for all n. Since $\langle Ae_n, e_n \rightarrow 0$, K is compact. Obviously,

$T = A - K$ is zero diagonal.

$(j) \Leftrightarrow (k) \Leftrightarrow (b)$

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