CHARACTERIZATION OF SCHWARZ NORMS IN BANACH SPACES

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Abstract

In this paper, we characterize Schwarz norms in Banach spaces. We give new results on the s-norms in B(H).

1 Introduction

Suppose that f is an analytic function in the open unit disk $U = \{z \in \mathbb{C} : |z|<1\}$ and is bounded i.e. $\|f\|_\infty = \sup\{|f(z)| : z \in U\} < \infty$. If f has the following additional properties, $f(0) = 0$, $\|f\|_\infty < 1$, then the following (Schwarz Lemma) holds:

If f is analytic in the open unit disk as described above and,

(i.) $|f(z)| \leq |z|$, $z \in U$

(ii.) $|f'(0)| \leq 1$,

and if the equality appears in (i) for one $z \in U - \{0\}$, then $f(z) = az$, where a is a complex constant with $|a| = 1$ and also if the equality appears in(ii), f behaves similarly. In case of operators, we have that, if $|T| \leq 1$, then $|f(T)| \leq \|f\|$ for each $f \in R(D)$ such that $f(0) = 0$. Here R(D) is the (sup-norm) algebra of the rational functions with no poles in the closed unit disk D and f(T) defined by the usual Cauchy integral around a circle slightly larger than the unit circle.[5] We note here that a contraction (i.e an operator T such that $\|T\| < 1$) $T \in B(H)$ has some relation with the closed unit disk of the complex plane, say for any contraction T and any complex-valued function f(z) defined and analytic on the closed unit disk ,then by von Neumann [9],[11] the norm equality holds; $\|f(T)\| \leq \|f\|_\infty \equiv \max_{|z|\leq 1}|f(z)|$where the operator f(T) is defined by the usual functional calculus[10]. The above lemma has an interesting application in the theory of operators namely the following assertions hold , if f is analytic in the open unit disk andf(0) = 0 with $\|f\|_\infty < 1$,then for any operatorT∈B(H), $\|T\| < 1$,(Berger and Stampfli) [2] we have $\|f(T)\| < \|T\|$.Clearly if we have an equality for some T, then f is of the formf(z) = az.Where a is a complex constant with $|a| = 1$. A norm , say, $\|T\| < 1$ on the algebra B(H) of all bounded operators T , is called a Schwarz norm if it is equivalent to the usual norm $\|.\|$and the Schwarz lemma holds for it,i.e for any f analytic in the open unit disc U with f(0) = 0 and $\|f\|_\infty < 1$, and for any T∈B(H), $\|T\| < 1$, we have $\|T\|^* < 1$

2 Preliminaries

We will in this section give the definitions that will be essential in our study. In the following **K** = **R** or **C**

Definition 2.1. For a set of points X,the pair (X; K) is called a linear space if for all x,y∈X andα, β ∈**K** then$\alpha x + \beta y \in X$

In case **K** = **R** then the pair is referred to as real linear space but if **K** = **C** then it is a complex linear space.

Definition 2.2. Let (X; **K**) be a linear space as defined above.A mapping $\|.\|: X \to \mathbb{R}$ is called a norm on X if it satisfies the following properties (norm axioms);

(i) $\|x\| \geq 0$ for all x∈X (non-negativity)

(ii) If x∈X and $\|x\| = 0$, then x = 0 (zero axiom)

(iii) $\|\lambda x\| = |\lambda|\|x\|$ for all x∈X and y∈K (homogenity)

(iv) $\|x + y\| \leq \|x\| + \|y\|\ \forall y, z \in X$ (triangular inequality)

The ordered pair (X; $\|:\|$) is called a normed linear space (n.l.s) over **K**

Definition 2.3. Suppose property number (ii) (zero axiom) in the above definition fails ,i.e if $x \in X$ and $\|x\| = 0$; $x = 0$, then the function , $\|:\|: X \to \mathbb{R}$

is referred to as seminorm on X.

Definition 2.4. Let (X,**K**) be a linear space and $\|:\|_1, \|:\|_2$ be norms on X we say that $\|:\|_1$ and $\|:\|_2$ are equivalent if their exists positive reals $,\beta$, such that

$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\| \; \forall x \in X$.The two norms generate the same open sets (same topology)

Definition 2.5. A sequence (x_n) is said to converge strongly in a normed linear space (X,$\|:\|$) if their exists $x \in X$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$

Definition 2.6. Let (X,$\|:\|$) be a normed linear space and ρ be the metric induced by $\|:\|$. If (X,ρ) is a complete metric , then we call (X,$\|:\|$) a Banach space or strongly complete normed linear space.(A normed linear space (X; $\|:\|$) is a Banach space if every strong Cauchy sequence of elements of X converges strongly in X)

Definition 2.7. Let (X,**K**) be a linear space. If M is a subset of X such that $x, y \in M$ and

$\alpha, \beta \in \boldsymbol{K} \to \alpha x + \beta y \in M,$ then M is called a subspace of X

Definition 2.8. Let X be a linear space over **K** and <,> : X↔**K** be a function with,

(i) $<x, x> \geq 0$ for all $x \in X$

(ii) $<x, x> = 0 \to x = 0$

(iii) $<y, x> = <x,y>^*$ or <x,y if **K** = $\mathbb{C}$ or K = $\mathbb{R}$respectively for all $x, y \in X$. where $<x,y>^*$ denotes the conjugate of the complex number <x,y>.

(iv) $<\lambda x, y> = \lambda<x, y>$ for all $x, y \in X$ and all $\lambda \in K$.

(v) $<x + y, z> = <x, z> + <y, z>$ for all $x, y, z \in X$ The function <.> is called inner-product (i.p) function and the real or complex number

<x, y>is called the inner product of x and y (in this order). The ordered pair (X,<.>) is called an inner product space or pre-Hilbert space over K. Let (X,<.>) be an inner-product space. The norm in X is given by $\|x\| = \sqrt{<x,x>}$ for all x$\in$ X and is called the norm determined by (or induced by) the inner-product function of x. The metric ρ determined by this norm $\|.\|$ as defined above is $\rho(x, y) = \|x - y\|$ for all x, y$\in$X is called the metric induced by the inner-product function<.>. If with respect to this norm $\|x\|$, defined above, (X,$\|.\|$) is strongly complete i.e(X,$\|.\|$) is a Banach space,then we refer to (X,$\|.\|$) as a Hilbert space i.e a Hilbert space is a complete inner product space.

Definition 2.9. Let H be a complex Hilbert space and T be a linear operator from H to H. T is said to be positive if<Tx, x>≥0, for all x in H. This can be denoted by T≥0 or 0≤T. T is said to be strictly positive or positive definite if<Tx, x> >0 for all x$\in H\backslash\{O\}$

Definition 2.10. If T$\in$B(H), then the operator T^{*}: H→H defined by <Tx, y> = <x, T^{*}y>for all x,y in H is called the adjoint of T. (T^{*} is also in B(H)) and $\|T^{*}\| = \|T\|$

Definition 2.11. An operator T$\in$B(H) is said to be self – adjoint if T^{*} = T and if T is linear on a linear subspace M of a Hilbert space H into M, then it is said t be Hermitian addition $\langle Tx, y\rangle = \langle x, Ty\rangle \forall x, y \in M$

Defination 2.12. Let H be a complete Hilbert space and T$\in$B(H). Then there exist unique self-adjoint operators A,B$\in$B(H) such that T=A + iB, A and B are given by A=$\frac{1}{2}(T+T^{*})$, B=$\frac{1}{2i}(T-T^{*})$ so that A is called real part of T denoted by ReT and B the imaginary part of T denoted by ImT. Note that $\mathrm{Re}\langle Tx,x\rangle = \langle(\mathrm{Re}T)x,x\rangle$ for every x∈ H. Indeed $\langle Tx,x\rangle = \frac{1}{2}\langle(T+T^*)x,x\rangle + i\frac{1}{2}\left\langle\left(\left(\frac{T-T^*}{2}\right)x,x\right\rangle\right.$, being a complex number, we have $\langle Tx,x\rangle = a$ + ib, where a, b are real numbers given by a= $\langle(\mathrm{Re}T)x,x\rangle$, b= $\langle(\mathrm{Im}T)x,x\rangle$

Defination 2.13. Let H be a complex Hilbert space and T∈ $B(H)$, The numerical range of T is the set W(T)⊂ℂ defined by W(T) = $\langle Tx,x\rangle : x\in H$ and $\|x\|=1\}$

3. Main Results

It is quite natural to investigate the problem about the existence of Schwarz norms on the algebra B(X) of all bounded operators on a Banach space X. For this we recall that a function [.] on X xX intoℂ is called a semi-inner product if the following conditions are satisfied:

1. $[x_1 + x_2, y] = [x_1, y] + [x_2, y]$
2. $[ax, by] = ab^* [x, y]$
3. $|[x, y]| \leq \|x\|.\|y\|$
4. $[x,x] > 0$ for $x \neq \bar{0}$ for all $x_1, x_2, x, y \in X$ and a, b are complex numbers.

Theorem 3.1. On every Banach space there exist a semi-inner product [,] with the property[x, x] = $\|x\|^2$(i.e it is compatible with the norm).Indeed for any $x \in X$ we define the functional $f_x \in X^2$. (Where X denotes the space of all the bounded functionals on X)with the properties;

(i) $\|f_x\| = \|x\|$

(ii) $f_x(x) = \|x\|^2$

The existence of the functional is guaranteed by Hahn-Banach theorem and we define

$[x, y] = f_y(x)$ and $f_{\lambda x} = \lambda^* f_x$which satisfy the four conditions above, for each λ∈ℂ, x ∈X. An operator T $\in$B(X) is called hermitian if $\| e^{iT} \| = 1$ for all real numbers t or equivalently,Bonsall[6] if $W(T) = \{[Tx,x] : \|x\| = 1\}$ is a subset of real numbers.

An operator T $\in$B(X) is called positive if T is hermitian and the spectrum of T is in the subset $\{x\in R : x > 0\}$.

Now the definition of the class S_Q can be as follows.

Definition 3.2. An operator T 2 S_Q if and only if

1. $\delta(T) \subset U$

2. For any $x\in X$, and $|z| <1$ $Re[(I + \sum Q^{\frac{1}{2}} T^n Q^{\frac{1}{2}} z^n)x,x] \geq 0$ where Q is a hermitian operator such that $Q^{1/2}$ is also a hermitian operator.The following results give indications about the possible existence of Schwarz norms.

Theorem 3.3. There exists a Banach spaceX and an operator T such thatRe[T x,x] $\geq$0does not implyRe[T^{-1} x, x] $\geq$ 0.

As an example to illustrate this,we consider the Banach space $\ell^p{}_2$ of all pairs x = (x_1,x_2) with the normx$\mapsto \|x\|_p = \{|x_1|^p + |x_2|^p\}^{\frac{1}{p}}$ $1 < p < \infty$.

In this case it can be seen that the semi-inner product compatible with the norm $[x, x] = \|x\|^2_p$ is given by$[x, y] = x_1|y_1|^{p-1} + x_2|y_2|^{p-2}$where x = (x_1, x_2) and y = (y_1, y_2) We consider an operator on this space with the matrix

$\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$where the elements a; b; c are complex numbers. We need to find conditions for the a, b ,c such that Re[T x, x] $\geq$0.A straight forward but complicated computation shows that these are :

1. Rea ≥ 0, Reb≥ 0

2. $|c| \leq (pRea)^{1/p} (qReb)^{1/q}$ $(1/p + 1/q = 1)$ and condition for $Re[T^{-1}x,x] \geq 0$ is

   $|\frac{c}{ab}| \geq (pRea^{-1})^{1/p}(qReb^{-1})^{1/q}$ and thus if $|c| \leq |a|^{1-2/p}(Repa)^{1/p}(Reqb)^{1/q}$ and this gives that

   $Re[Tx, x] \geq 0$.

Remark 3.4. In case of Hilbert space (and invertible) operators, the condition , $ReT \geq 0$ implies the condition $ReT^{-1} \geq 0$

We now give an example of a Banach space with the property that the induced norm on B(X) is not a Schwarz norm.

Example 3.5 If $X = \ell^1{}_2$ then the induced norm on B(X) is not a Schwarz norm.We consider the operator T with the matrix(triangular) $\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$ and a simple computation shows that $\|T\| = \max\{|a| + |c|, |b|\}$. We now take 0<a<1 and in this case the operator with the matrix $\begin{bmatrix} a & 0 \\ 1-a & 1 \end{bmatrix}$ is a contraction operator. An elementary computation shows that |a|<1, the conformal map/function $\vartheta\alpha(z) = (z-\alpha)(1-\overline{\alpha}z)^{-1}$ for all $z \in \mathbb{C}$, take contractions; now consider the function $f\alpha(T) = (1-\overline{\alpha}T)^{-1}(T-\alpha 1)$. The computation of the norm of the operator $f\alpha(T)$ shows that this is given by $\|f\alpha(T)\| = a|\alpha + a + (1-a)|\frac{1+\alpha+1+\overline{\alpha}}{(1+\overline{\alpha a})(1+\alpha)}|$ and thus for $\|f\alpha(T)\| \leq 1$, where α is a real number , we obtain $a|\alpha + a| + (1-a)(1+a) \leq |1+\alpha a$ which is not $\alpha = -1/2(a+1)$. In view of the results is of interest.

Proposition 3.6 If X is a complex Banach space and for any contraction T, f(T) is also a contraction for all $|f| \leq 1$, then X is a Hilbert space .

Proof:

Let $x_0 \in X$ be arbitrary $x_0 \in X$ such that $\|x_0\| \|x^*_0\| \leq 1$and define the operator on X by the relation $T_x = x_0^*(x)x_0$. Its clear that T is a contraction . From the hypothesis it follows that $\|x_0^*(x)x_0 + x \leq \|x + \alpha^* x^*(x)x_0\|$. Now if $x,y \in X$ and $\|x\| \geq \|y\| \geq 0$, we obtain from the H-Banach theorem that there exists $x_0^* \in X^*$ such that $\|x_0^*\| = \|x\|^{-1}$, $x_0^*(x)=1$.

We take $x_0 = y$ and remark that the operator T constructed with these element gives us $\|y + \alpha x\| \leq \|x + \alpha^* y\| < 1$ and from the continuity argument, it follows that this relation holds for $|\alpha^*|=$

1. Now if $\|x\| = \|y$, changing the role of x with y and α with α^*, we obtain$\| \leq \|x + \alpha^* y\| \geq \|y + \alpha x\|$, thus we have the equality$\|x + \alpha^* y\| = \|y + \alpha x\|$. Now if $|\alpha|>1$, then for $\beta = \frac{1}{\alpha}$ we have by the above result$\|x + \alpha^* y\| = |\alpha \| \beta x + y| = |\alpha| \|x + \beta y\| = \|\alpha x + y\|$ and

thus the relation is true for any α Now for α =p/q, p and q being real numbers we obtain thatp/q, p and q being real numbers we obtain that $\|px + qy\| = |q| \|p/qt + x\| = |q| \|p/qt + x\| = |q| \|y + p/qx\| = \|qy + px\|$and thus for any x and y , $\|x\| = \|y\|$ and any p,q real numbers we obtain that $\|px + qy\| = \|qx + py\|$ and by a famous result of F.A.Ficken,this relation is characteristic for a norm to be inner product norm,i.e,there exists an inner product $\langle . \rangle$ on X such that for all $x \in X$, $\|X\|^2 = \langle x, x \rangle$

4 Conclusion

A Schwarz norm can be constructed from the sum of a norm and a seminorm and that Schwarz norms are easily realizable in the Hilbert space context.

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